A New Perspective on Kesten's School Choice with Consent Idea*

Qianfeng Tang† and Jingsheng Yu

School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China

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Abstract

We revisit the school choice problem with consent proposed by Kesten (2010), which seeks to improve the efficiency of the student-optimal deferred acceptance algorithm (DA) by obtaining students' consent to give up their priorities. We observe that for students to consent, we should use their consent only when their assignments are Pareto unimprovable. Inspired by this perspective, we propose a new algorithm which iteratively reruns DA after removing students who have been matched with underdemanded schools, together with their assignments. While this algorithm is outcome equivalent to Kesten's EADAM, it is more accessible to practitioners due to its computational simplicity and transparency on consenting incentives. We also

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†Corresponding author.

Email addresses: tangqianfeng198@gmail.com (Q. Tang), jingshengyu1987@gmail.com (J. Yu).

adapt this algorithm for school choice problems with weak priorities to simplify the stable improvement cycles algorithm proposed by Erdil and Ergin (2008).

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1 Introduction

In a school choice problem, students have preferences for schools and in turn, schools rank-list students by their priorities. An allocation mechanism matches students with seats at schools. The Gale and Shapley (1962) student-proposing deferred acceptance algorithm (henceforth, DA) always selects the optimal stable matching for students. Nevertheless, it is well known that the DA matching is not necessarily Pareto efficient. Abdulkadiroğlu, Pathak and Roth (2009), using NYC high school match data, show that in practice, such inefficiency is empirically significant. Kesten (2010) proposes the school choice problem with consent, which seeks to improve the efficiency of the DA algorithm by obtaining students' consent to give up their priorities. We revisit this problem and offer a new perspective.

As is well known in the literature (see e.g., Ergin, 2002, and Kesten, 2010), the inefficiency of DA may arise when certain cycles exist in schools' priority lists. Consequently, it may happen that during the DA algorithm procedure, some student i applies to school s and is tentatively accepted, but her tentative acceptance at s initiates a chain of rejections that eventually lead s to reject student i herself. By applying to school s, student i gains nothing, but potentially blocks trading among other students. In Kesten (2010), i is called an "interrupter" at s.

In a school choice problem with consent, some or all students consent to give up their priorities at schools that are better than their assignments. To improve the efficiency of the DA algorithm, Kesten designs the efficiency-adjusted DA mechanism (henceforth, EADAM), which iteratively reruns DA after removing the last interruptions caused by consenting interrupters in the DA procedure. He then shows that no student has incentive to not consent under EADAM, and when all students consent, EADAM is Pareto efficient.

We take a new perspective on school choice with consent by directly examining consenting incentives. We observe that to make sure that students do not have incentives to not consent, we should use their consent only when they are (Pareto) unimprovable, so that their consent won't hurt their opportunities of being improved to better schools. This perspective brings us transparency in consenting incentives and makes the algorithms designed or interpreted based on this perspective more accessible to practitioners. For a given matching, to identify unimprovable students in a convenient way, we define underdemanded schools. We say that a school is underdemanded at a matching if no student prefers it to her assignment. Since Pareto improvements of any non-wasteful matching must take the form of trading cycles, students matched at underdemanded schools at the DA matching are all unimprovable. Moreover, a school is underdemanded at the DA matching if and only if it never rejects any student during the DA procedure.

By focusing on unimprovable students, we propose a new algorithm–the simplified EADAM–for school choice problems with consent. If all students consent, this algorithm starts by running DA, and then iteratively reruns DA after removing students matched with underdemanded schools together with their assignments. If not all students consent, whenever we remove a non-consenting student, for each remaining school that she desires, we also make sure that the remaining students who have lower priorities than her are unacceptable to this school. We show that in each round, there exists at least one underdemanded school; therefore, at least one school will be removed. As a result, this algorithm stops within $m + 2$ rounds if there are m schools. We also show that the simplified EADAM is Pareto efficient when all students consent and is constrained efficient otherwise. A matching is constrained efficient if it does not violate the priorities of non-consenting students, but any matching that Pareto dominates it does.

Although the simplified EADAM and Kesten's EADAM differ in several ways, they share the same iterative structure and, more fundamentally, they can be unified under the perspective of focusing on unimprovable students. To show the latter, we prove in a lemma that the lastly rejected interrupters of the DA procedure are all matched with essentially underdemanded schools and hence are unimprovable at the DA matching. Therefore, under both mechanisms, even if a student consents, her consent will be used only after her assignment becomes unimprovable. Consequently, her consent decision can only affect other students' assignments, but not her own assignment. This argument makes the mechanisms' consenting incentives transparent and renders them more accessible to practitioners. We then show that the two mechanisms are outcome equivalent and that this equivalence holds more generally among mechanisms designed by focusing on unimprovable students.

We also apply the simplified EADAM to school choice with weak priorities, following the works of Erdil and Ergin (2008) and Kesten (2010). The goal is to recover the efficiency loss of DA caused by fixed tie-breaking. We begin by transforming the problem by assuming that no student consents to give up priorities, except at tied priorities. In the adaptation of the simplified EADAM, we iteratively rerun DA after removing students matched at underdemanded schools and making them yield tied priorities to the remaining students. This adapted algorithm can be viewed as a stable improvement cycles mechanism proposed by Erdil and Ergin, but with endogenous cycle selection. Since this algorithm recovers all efficiency loss caused by the inappropriate tie-breaking of each round's removed students at once–and because at least one school is removed in each round–it stops very quickly.

Our contribution is to propose a perspective on designing for school choice with consent, and to design new mechanisms and interpret existing mechanisms based on that perspective. Bando (2014) shows that the EADAM outcome can be supported by a strictly strong Nash equilibrium of the preference revelation game under DA, and along the way independently proposes another simplification of Kesten's EADAM. Bando's algorithm focuses on the removal of the last-step proposers of the DA algorithm, instead of the lastly rejected interrupters. Since last-step proposers are unimprovable, his approach can also be unified under our perspective of focusing on unimprovable students. Kesten and Kurino (2013) define underdemanded schools in the same way as we do and they are the first to introduce this concept; they also study some general properties of this concept. However, they have a different motivation in mind–by restricting the preference domain, they try to resolve the trade-off between strategy-proofness and Pareto efficiency of DA. Our paper also relates to other literature that studies the inefficiency of DA; for example, Kesten (2006), Abdulkadiroğlu, Pathak and Roth (2009) and Erdil (2014), among others. Kesten and Kurino offer a detailed review of the literature on the trade-off between strategy-proofness and efficiency in improving on DA.

This paper is organized as follows. We introduce the basic model and Kesten's EADAM in Section 2 and define underdemanded schools and the simplified EADAM in Section 3. In Section 4, we present our main results and the application of the simplified EADAM on weak priorities. Section 5 concludes. All proofs are relegated to the Appendix.

2 Preliminaries

2.1 The model

A set of students $I = \{i_1, i_2, \ldots, i_n\}$ are to be matched with seats at a set of schools $S = \{s_1, s_2, \ldots, s_m\}$. Let i denote a generic element of I and let s denote a generic element of S. Let q_s denote the capacity (or quota) of school s. If a student is unassigned, we say that she is matched with the null school $\emptyset$, which has unlimited capacity. For each agent $i \in I$, let P_i denote her strict preference over $S \cup \{\emptyset\}$ and let R_i be the symmetric extension of P_i. If student i prefers (weakly prefers) school s to school s', we write sP_is' (sR_is', resp.). Let $\succ_s$ denote the strict priority list at school s. If student i has higher priority than student j at school s, we write $i \succ_s j$. We assume that to each school, all students are acceptable.

A **school choice problem** consists of a pair $(P, \succ)$, where $P \equiv (P_i)_{i \in I}$ and $\succ \equiv (\succ_s)_{s \in S}$ are the preference profile and priority profile, respectively.[1] A **matching** μ is a mapping from I to $S \cup \{\emptyset\}$ such that $|\mu^{-1}(s)| \leq q_s, \forall s \in S$. Matching ν **Pareto dominates** matching μ under preference profile P if for all $i \in I, \nu(i)R_i\mu(i)$, and for some $j \in I, \nu(j)P_j\mu(j)$. Matching ν **weakly Pareto dominates** matching μ under preference profile P if for all $i \in I, \nu(i)R_i\mu(i)$. A matching is **Pareto efficient** under preference profile P if it is not Pareto dominated by any other matching under P.

Student i **desires** school s at matching μ if $sP_i\mu(i)$. We say that at matching μ, student j **violates student i's priority** at school s if i desires s, $\mu(j) = s$ and $i \succ_s j$. A matching μ is **fair** (Balinski and Sönmez, 1999) if no student's priority at any school is violated at μ, and it is **non-wasteful** if any school $s \in S \cup \{\emptyset\}$ that is desired by some student at μ satisfies $|\mu^{-1}(s)| = q_s$. A matching μ is **stable** if it is fair and non-wasteful.

[1] To describe a school choice problem more formally, we should also include the profile of capacities $q = (q_s)_{s \in S}$. Nevertheless, we suppress this for notational simplicity.

An **improvement cycle** at μ consists of an ordered list of students $i_1, i_2, \ldots, i_K = i_1$ such that for each $1 \leq k < K$, student i_k desires school $\mu(i_{k+1})$. School s is **unacceptable (acceptable)** to student i if $\emptyset P_i s$ ($s P_i \emptyset$). An **allocation mechanism** φ selects a matching $\varphi(P, \succ)$ for every problem $(P, \succ)$. An allocation mechanism is Pareto efficient if $\varphi(P, \succ)$ is Pareto efficient for every $(P, \succ)$, and is stable if $\varphi(P, \succ)$ is stable for every $(P, \succ)$.

2.2 Kesten's EADAM

The benchmark algorithm we study is the famous **Gale-Shapley student-proposing deferred acceptance algorithm** (DA) proposed by Gale and Shapley (1962). DA is an allocation mechanism that is optimally stable (Gale and Shapley, 1962) and strategy-proof (Dubins and Freedman, 1981; Roth, 1982). For each school choice problem $(P, \succ)$, DA operates as follows:

Step 1 Each student applies to her most desirable school. Each school tentatively accepts the best students according to its priority list, up to its capacity, and rejects the rest.

Step $k, k \geq 2$ Each student rejected in the previous round applies to her next best school. Each school that faces new applicants tentatively accepts the best students according to its priority list, up to its capacity, and rejects the rest, among both new applicants and previously accepted students.

The algorithm stops when no student is rejected.

DA matches each student with the last school that accepted her during the algorithm. We denote the DA matching for problem $(P, \succ)$ by $DA(P, \succ)$ and denote student i's DA assignment by $DA(P, \succ)(i)$.

Although DA produces student-optimal stable matching, the DA matching is not necessarily Pareto efficient for students, and to Pareto improve on DA, stability has to be relaxed.[2] In a pioneering work, Kesten (2010) proposes the **school choice problem with consent**, in which each student is asked whether she consents to give up her priorities at desired schools–or equivalently, whether she allows other students to violate her priorities–if doing so does not hurt her own assignment but potentially improves the assignments of other students.

To implement the improvements on consent, Kesten proposes the **efficiency-adjusted deferred acceptance mechanism (EADAM)**, which focuses mainly on interrupters. Formally, in the DA procedure of a school choice problem, if student i is tentatively accepted by school s at some step t and is later rejected by school s at some step $t' > t$, and at least one other student is rejected by school s at some step l such that $t \leq l < t'$, then student i is an **interrupter** for school s and the pair (i, s) is an interrupting pair of step t'.

For any school choice problem $(P, \succ)$ with a fixed set of consenting students, Kesten's EADAM operates as follows:

Round 0 Run DA for the problem $(P, \succ)$.

Round $k, k \geq 1$ Identify the last step of the round-$(k-1)$ DA procedure in which consenting interrupter(s) are rejected, and then identify all interrupting pairs of this step that contain a consenting interrupter and, for each pair, remove the respective school from the interrupter's preference. After that, rerun DA (round-k DA) with the new preference profile.

Stop when there are no more consenting interrupters.

Kesten (2010) shows that when all students consent, EADAM is Pareto efficient.

[2]See Example 1 for a simple illustration.

3 The simplified EADAM

3.1 Underdemanded schools

Following Kesten (2010), we revisit how to design a mechanism to Pareto improve the efficiency of DA matching through obtaining consent from students. The approach we take is to directly examine consenting incentives. To make sure that students do not have incentives to not consent, the mechanism must not: (i) hurt any student's assignment comparing with her DA assignment; or (ii) hurt the improvement opportunity of any consenting student. The inspiration we receive from the latter point is that in the algorithm, we should use the consent of a student only when her assignment is not Pareto improvable anymore. Before we unify Kesten's EADAM and Bando (2014)'s modification under this perspective, we first study how this new perspective can improve the algorithm's design.

We now define underdemanded schools, which will play a crucial role.

Definition 1. *A school s is **underdemanded** at a matching μ if no student prefers s to her assignment under μ.*

It is straightforward to see that a school is underdemanded at the DA matching if and only if it never rejects any student throughout the DA procedure.

The concept of underdemanded schools can also be generalized through the following recursive construction. We say that a school is **tier-**0 **underdemanded** at matching μ if it is underdemanded at μ, and for any integer $k > 0$, a school is **tier-**k **underdemanded** at matching μ if (i) it is desired only by students matched with lower-tier underdemanded schools at μ, and (ii) it is desired by at least one of the students matched with tier-$(k-1)$ underdemanded schools at μ.

Definition 2. *School s is **essentially underdemanded** at matching μ if it is tier-k underdemanded at μ for some integer $k \geq 0$.*

The set of essentially underdemanded schools at the DA matching can also be identified through a recursive process, by reviewing the DA procedure that produces this DA matching. Tier-0 underdemanded schools are the schools that never reject any student throughout the DA procedure. After removing tier-0 underdemanded schools and the students matched with them, tier-1 underdemanded schools are the remaining schools that never reject any remaining students throughout the DA procedure, and so on. We now present a simple illustration of these concepts.

Example 1. This example is adapted from Ergin (2002). Consider a school choice problem $(P, \succ)$ in which there are four schools, s_1, s_2, s_3 and s_4, each with one seat, and four students, i_1, i_2, i_3, and i_4. The tables below, from left to right, illustrate the schools' priorities, the students' preferences, and the DA procedure, respectively.

s_1	s_2	s_3	s_4
i_1	i_3	i_2	i_4
i_2	i_1	i_4	$\vdots$
i_3	$\vdots$	$\vdots$	
$\vdots$			

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_2	s_1	s_1	s_3
s_1	s_3	s_2	s_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Step	s_1	s_2	s_3	s_4
1	$\boxed{i_2}, i_3$	$\boxed{i_1}$	$\boxed{i_4}$	$\vdots$
2	$\vdots$	$\boxed{i_3}, i_1$	$\vdots$	
3	$i_2, \boxed{i_1}$	$\vdots$		
4	$\vdots$		$\boxed{i_2}, i_4$	
5	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_2}$	$\boxed{i_4}$

During the DA procedure, student i_2 is the only interrupter and she interrupts school s_1. Effectively, the interruption of i_2 blocks a trading between i_1 and i_3 and leads to the inefficiency of the DA matching $DA(P, \succ)$.

At $DA(P, \succ)$, both s_3 and s_4 are essentially underdemanded. Specifically, s_4 is (tier-

0) underdemanded, while s_3 is tier-1 underdemanded. Note that the (lastly rejected) interrupter i_2 is matched with an essentially underdemanded school at the DA matching.

The non-wastefulness of DA implies that every Pareto improvement on the DA matching can only be implemented through trading. If a student is matched with an essentially underdemanded school at the DA matching, then–since she cannot be part of any trading cycle–her assignment has to remain unchanged in any Pareto improvement on the DA matching.

We say that the assignment of student i is **not Pareto improvable** (or, simply, unimprovable) at $DA(P, \succ)$ if for every matching μ that Pareto dominates $DA(P, \succ), \mu(i) = DA(P, \succ)(i)$.

Lemma 1. *At the DA matching, all students matched with essentially underdemanded schools are not Pareto improvable.*

Therefore, the concept of (essentially) underdemanded schools offers us a convenient way to identify a large set of unimprovable students. The lemma above still holds if the DA matching is replaced with any non-wasteful matching; we prove this in the Appendix. Kesten and Kurino (2013) present the same result for underdemanded schools. Similar results are also presented in Abdulkadiroğlu, Pathak and Roth (2009) and Erdil (2014), who show that any Pareto improvement over a non-wasteful matching must be in the form of a reshuffling of objects among matched agents.

3.2 The simplified EADAM

Consider any school choice problem $(P, \succ)$ with a fixed set of consenting students. We propose **the simplified efficiency-adjusted deferred acceptance mechanism (the simpli-**

fied EADAM), which operates as follows:

Round 0 Run DA for the problem $(P, \succ)$.

Round $k, k \geq 1$ This round consists of three steps:

1. Identify the schools that are underdemanded at the round-$(k-1)$ DA matching, settle the matching at these schools, and remove these schools and the students matched with them.[3]
2. For each removed student i who does not consent, each remaining school s that student i desires and each remaining student j such that $i \succ_s j$, remove s from j's preference.[4]
3. Rerun DA (the round-k DA) for the subproblem that consists of only the remaining schools and students.

Stop when all schools are removed.

The simplified EADAM preserves the iterative structure of Kesten's EADAM, while taking a new perspective by focusing on unimprovable students instead of (only) interrupters. The new perspective leads to several differences. First, at the end of each round, we remove all students matched with underdemanded schools, and thereby remove all of their desired applications instead of removing only the last interruption. Second, after the removal of non-consenting students–since we already know which matchings among the remaining schools and students would violate their priorities–we modify the preferences of the remaining students accordingly to avoid violations of their priorities in future rounds of the algorithm.

[3]Equivalently, we can keep these students and modify their preferences so that for each, only the underdemanded school that she is matched with is acceptable to her.

[4]In practice, we can simply remove student i and all students ranked below i at $\succ_s$ from $\succ_s$, and during the DA procedure, let a student skip s and apply directly to the next best school if she is not listed by $\succ_s$.

We now present some preliminary results for the simplified EADAM, followed by examples of the algorithm. Main results of the algorithm are presented in the next section.

Lemma 2. *For each $k \geq 1$, the round-k DA matching of the simplified EADAM weakly Pareto dominates that of round-$(k-1)$.*

In particular, the simplified EADAM weakly Pareto dominates the DA algorithm. As a result of Lemma 2, at the DA matching produced in each round of the simplified EADAM, there exists at least one underdemanded school. Specifically, the null school is always underdemanded at the round-0 DA matching. For the DA matching of any later round, no student is unmatched, and any school that accepts students in the last step of that round's DA procedure is underdemanded.

Proposition 1. *The simplified EADAM is well-defined and stops within $|S \cup \{\emptyset\}| + 1 = m + 2$ rounds.*

A nice feature is that this upper bound of number of rounds needed in running the algorithm does not depend on the number of students or the number of available seats. We can also enhance the simplified EADAM slightly by replacing underdemanded schools with essentially underdemanded schools in its definition–doing so potentially reduces the number of rounds needed in running the algorithm.

3.3 Examples

To apply the simplified EADAM, we revisit example 3 of Kesten (2010). In the problem $(P, \succ)$, there are five schools $\{s_1, \ldots, s_5\}$, where s_5 has two seats and other schools each has one seat. There are six students. The priority profile $\succ$ and preference profile P are described in the tables below.

s_1	s_2	s_3	s_4	s_5
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	
i_5	i_4	i_2	i_6	
i_6	i_1	i_3	$\vdots$	
i_4	$\vdots$	$\vdots$		
i_3				

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_3	s_1	s_1	s_4
s_1	s_1	s_4	s_2	s_5	s_1
s_3	s_5	s_2	s_4	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$			s_2
					s_5

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As in Kesten (2010), we illustrate the DA procedure of the problem (round-0 DA of the simplified EADAM) in a table. We see that all students are interrupters.

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		$i_2, \boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}, i_1$		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}, i_6$		
7		$\boxed{i_6}, i_4$	$\vdots$		
8		$\vdots$		$\boxed{i_4}, i_3$	
9		$i_6, \boxed{i_3}$		$\vdots$	
10	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_5}, \boxed{i_6}$

Example 2 (All students consent)**.** Suppose all students consent and we run the simplified EADAM for the problem above. At the round-0 DA matching, s_5 is the only under-

demanded school and students i_5 and i_6 are matched with it. So in round-1 we remove s_5 together with i_5 and i_6, and rerun DA with the rest of the schools and students. The procedure of round-1 DA is illustrated in the following table:

Step	s_1	s_2	s_3	s_4
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	
2				$\boxed{i_3}$
3	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$

At the end of round-1, all schools are underdemanded except for s_3. So in round-2, we first remove all other schools and their matched students, and then run DA for s_3 and i_2. The round-2 DA is trivial and the algorithm stops immediately afterward. The final matching is the same as the round-1 DA matching.

If we run the simplified EADAM with essentially underdemanded schools instead, then since all schools are essentially underdemanded after running round-1 DA, we can stop the algorithm at that point.

Example 3 (Not all students consent)**.** Suppose only student i_6 does not consent. At the beginning of round-1, we still remove s_5 together with i_5 and i_6. Moreover, i_6 does not consent; desires s_1, s_2, s_3 and s_4; and has higher priority than i_3 and i_4 at s_1, than i_1 and i_4 at s_2 and than i_2 and i_3 at s_3. Therefore, before running round-1 DA, we remove s_1 from P_{i_3} and P_{i_4}, s_2 from P_{i_1} and P_{i_4}, and s_3 from P_{i_2} and P_{i_3}. The modified preference profile P^1

of the remaining students and the round-1 DA procedure are illustrated by

$P^1_{i_1}$	$P^1_{i_2}$	$P^1_{i_3}$	$P^1_{i_4}$
s_1	s_1	s_4	s_4
s_3	s_5	s_2	$\vdots$
$\vdots$	$\vdots$	$\vdots$	

Step	s_1	s_2	s_3	s_4
1	$i_1, \boxed{i_2}$			$i_3, \boxed{i_4}$
2		$\boxed{i_3}$	$\boxed{i_1}$	
3	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$

Again, we can now stop, since all schools are essentially underdemanded. We see that when i_6 does not consent, the algorithm generates the same matching as the original DA. That is, $DA(P, \succ)$ cannot be Pareto improved without violating the priorities of i_6.

Suppose, instead, only i_5 does not consent. After the removal, only s_1 will be removed from P_{i_3} and P_{i_4}. The new $P^{1\prime}$ and the DA procedure for round-1 are given by

$P^{1\prime}_{i_1}$	$P^{1\prime}_{i_2}$	$P^{1\prime}_{i_3}$	$P^{1\prime}_{i_4}$
s_2	s_3	s_3	s_2
s_1	s_1	s_4	s_4
s_3	s_5	s_2	$\vdots$
$\vdots$	$\vdots$	$\vdots$	

Step	s_1	s_2	s_3	s_4
1		$i_1, \boxed{i_4}$	$\boxed{i_2}, i_3$	
2	$\boxed{i_1}$			$\boxed{i_3}$
3	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$

We observe that i_1 and i_4 can still benefit from trading with each other. However, such trading is not implemented, as doing so would violate the priority of i_5.

4 Results and application

4.1 Main results

The first objective of designing for school choice with consent is to achieve constrained efficiency. Let the set of non-consenting students in a school choice problem be fixed. We say that a matching μ is **constrained efficient** if μ does not violate any non-consenting student's priority, but any matching which Pareto dominates μ does. An allocation mechanism φ is constrained efficient if $\varphi(P, \succ)$ is constrained efficient for each school choice problem $(P, \succ)$ with consent.

Theorem 1. *The simplified EADAM is Pareto efficient when all students consent and is constrained efficient otherwise.*

This theorem extends Theorem 1 of Kesten (2010) by providing the result for constrained efficiency for problems with non-consenting students. The simplification of the algorithm also brings us a slightly more straightforward intuition that underlies (constrained) efficiency. Round by round, unimprovable consenting students help improve the assignments of others as much as they can conditional on not hurting their own assignments, which renders new students unimprovable. Naturally, when the algorithm stops, no student's assignment can be further improved without hurting the assignments of others or violating the priorities of non-consenting students.

The second objective of designing for school choice with consent is to ensure that students do not have incentives to not consent. We show that the following result (Kesten, 2010, Proposition 3) also holds for the simplified EADAM.

Theorem 2. *Under the simplified EADAM, the assignment of any student does not change whether she consents or not.*

Instead of providing a proof for this theorem, we only need to note that: (i) given the definition of the simplified EADAM, even if a student consents, her consent will be used only if her assignment is not Pareto improvable–that is, only if her assignment is settled; and (ii) whether a student consents or not does not change the rounds of the algorithm that take place before she becomes unimprovable. As a result, the consent of any student affects only other students' assignments, but not her own. Therefore, this algorithm fully resolves potential conflicts between helping others and being helped by them.

Some deep discussions are used in Kesten (2010) to show that Kesten's EADAM satisfies consenting incentives. In fact, this result is also transparent, once we see that like the simplified algorithm, Kesten's EADAM also focuses on unimprovable students.

Lemma 3. *The lastly rejected interrupters of the DA procedure are matched with essentially underdemanded schools at the DA matching.*

We relegate the more general result to the Appendix, which states that if step t is the last step in which consenting interrupters are rejected, and $DA(P, \succ)(i)$ accepts new students after step t, then student i is not improvable at $DA(P, \succ)$ without violating the priorities of non-consenting students.

The unification of mechanisms under the same perspective naturally leads to their outcome equivalence.

Theorem 3. *For every school choice problem with consent, the simplified EADAM produces the same matching as Kesten's EADAM does.*

Since the proof of this theorem relies only on the unimprovability of removed students, we can easily adapt it to show the outcome equivalence between Kesten's EADAM and similar algorithms that iteratively use the consent of unimprovable students'. Specifically, we can think of algorithms that focus on last-step proposers (Bando, 2014), students matched at essentially underdemanded schools, students matched at schools that settle after the step at which the last consenting interrupters are rejected, and so on.

The perspective of focusing on unimprovable students allows for transparency of consenting incentives, and therefore makes all of these algorithms more accessible to practitioners. When trying to obtain consent from students, policy makers need only ensure the students that their consent will be used only when their assignments become unimprovable–which, in turn, does not depend on their consenting decision. For simplicity (not only from the computational point of view), we recommend the simplified EADAM for practice.

We now illustrate with an example why a student may be reluctant to consent when an improvement mechanism removes the applications of students who are still Pareto improvable. In general, a student would be reluctant to consent–even when doing so will not directly hurt herself–if she sees the chance that if she consents her consent will be used to help others, but if she does not consent the consent of others might instead be used to help her.

Example 4. Let's revisit Example 2, where at the DA matching, the removal of i_5 and i_6 improves the assignments of all others. Now consider another improvement mechanism that intends to achieve constrained efficiency: Whenever there are top trading improvement cycles (defined as in Gale's TTC; see Shapley and Scarf, 1974) that do not violate non-consenting students' priorities, it implements such trading cycles first.

Suppose i_1 and i_3 do not consent and i_5 and i_6 consent. If, in addition, i_2 and i_4 consent, then since the top trading cycle $(i_1, s_3) \leftrightarrow (i_3, s_2)$ only violates the priorities of consenting students i_2, i_4, and i_6, it will be implemented. After that, the new matching obtained is Pareto efficient:[5]

s_1	s_2	s_3	s_4	s_5
i_2	i_1	i_3	i_4	i_5, i_6

However, if i_2 and i_4 do not consent, then to achieve constrained efficiency, the improvement mechanism can only remove the applications of i_5 and i_6 to help others. This time, the assignments of i_2 and i_4 will instead be improved.

4.2 Application on weak priorities

In practice, schools often have weak priorities over students, and in such cases a single tie-breaker is applied to school priorities before running DA. Erdil and Ergin (2008) show that the matching produced by DA with single tie-breaking can be improved on without violating priorities, and propose the use of stable improvement cycles algorithm to accomplish this.

Kesten (2010) shows that the same improvement can be made through an adaptation of EADAM. In this adaptation, after running DA in each round of EADAM, the last interrupters who are tied with other students at the interrupted school are identified. Ties are then re-broken in a way that does not favor these interrupters, which potentially prevents the blocking of cycles. With underdemanded schools in mind, after running DA in each round of the algorithm, we can re-break ties for all students matched with underdemanded schools so that they are unfavored. This is because Lemma 1 still holds. Therefore, such students still cannot be Pareto improved, and breaking ties not in favor

[5] For this specific school choice problem, this new matching is also selected by the top trading cycles mechanism (TTC) proposed by Abdulkadiroğlu and Sönmez (2003). Note that although TTC Pareto dominates DA in this example, this does not hold in general (see Kesten, 2006).

of them is equivalent to asking for their consent to locally yield to others at tied priorities.

Consider any school choice problem $(P, \succ)$ with weak priorities. To recover the efficiency loss from fixed tie-breaking, we adapt the simplified EADAM as follows:

Round 0 Randomly draw a tie-breaker to break ties in priority lists, and then run DA for the induced problem.

Round $k, k \geq 1$ This round consists of three steps:

1. Identify the schools that are underdemanded at the round-$(k-1)$ DA matching, settle the matching at these schools, and remove these schools and the students matched with them.
2. For each student i who is removed, each remaining school s that student i desires, and each remaining student j who ranks strictly lower than i at the original weak priority list $\succ_s$, remove s from j's preference. [6]
3. Rerun DA (the round-k DA) for the subproblem that consists of only the remaining schools and students.

Stop when all schools have been removed.

This adaptation of the simplified EADAM differs itself mainly in step 2 of each round $k, k \geq 1$. To ensure the stability of the final matching, we treat each student as a non-consenting student in the algorithm, but assume that when necessary, she has to consent to yield her priority to the others who have the same priority at every school. As usual, we use a student's consent only when her assignment cannot be further improved. For school choice problems with both weak priorities and consenting students,

[6]In practice, we can simply remove student i and all students ranking strictly lower than i at the original weak priority list $\succ_s$ from the induced priority list (obtained after tie-breaking); during the DA procedure, let a student skip s and apply directly to the next best school if she is not listed by the induced priority list.

the simplified EADAM can also be adapted accordingly. These adaptations of the simplified EADAM are also constrained efficient, following the same proof intuitions as that of Theorem 1.

We now compare the simplified EADAM with the stable improvement cycles (SIC) algorithm proposed by Erdil and Ergin (2008). First, the simplified EADAM implements stable improvement cycles. A special feature of it is that instead of identifying the SICs and then randomly selecting one to improve, it starts by relaxing the priorities of unimprovable students at tied priorities and then reruns DA to endogenously select the SIC. Essentially, the algorithm endogenously adjusts the tie-breaking. Second, in each round, the algorithm implements at once all stable improvements that can be obtained after the removal of students. Since in each round we remove all students matched with underdemanded schools, the algorithm also stops within $|S \cup \{\varnothing\}| + 1$ rounds.[7]

Example 5. Suppose $S = \{s_1, s_2, s_3, s_4\}, I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $q_s = 1, \forall s$. The weak priority profile $\succ$ and preference profile P are described by the tables below.

s_1	s_2	s_3	s_4
i_1	i_3	i_5	i_6
i_2, i_3	i_1	i_3	i_3
i_4, i_5	$\vdots$	i_2	i_2
$\vdots$		$\vdots$	$\vdots$

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_4	s_1	s_1	s_4
s_1	s_4	s_3	$\varnothing$	s_3	$\vdots$
$\vdots$	s_1	s_1		$\vdots$	
	$\varnothing$	s_2			
		$\vdots$			

Suppose the fixed tie-breaking rule is $i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5$. The DA procedure of

[7]See Erdil and Ergin (2008), p. 676, for the definition of the SIC algorithm and discussions of cycle selection and computational complexity.

the induced problem is as follows:

Step	s_1	s_2	s_3	s_4	$\emptyset$
1	$\boxed{i_4}, i_5$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_6}, i_3$	$\vdots$
2	$\vdots$	$\vdots$	$i_2, i_3, \boxed{i_5}$	$\vdots$	
3	$i_4, \boxed{i_3}$		$\vdots$	$\boxed{i_6}, i_2$	
4	$\boxed{i_2}, i_3$			$\vdots$	$\boxed{i_4}$
5	$\vdots$	$i_1, \boxed{i_3}$			$\vdots$
6	$i_2, \boxed{i_1}$	$\vdots$			
7	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_5}$	$\boxed{i_6}$	$\boxed{i_2}, \boxed{i_4}$

We focus on what happens at school s_1. Two interrupters, i_2 and i_4, interrupt school s_1; the only underdemanded school is the null school, which is also the school that the two interrupters are matched with.

We first remove the null school, together with i_2 and i_4. After that, since both i_2 and i_4 have tied priorities at s_1, we also need to remove s_1 from the preferences of any student who has strictly lower priority than i_2 or i_4 at $\succ_{s_1}$. We need only do this for i_5 and not for i_3. This is because $i_3 \sim_{s_1} i_2$ and $i_3 \succ_{s_1} i_4$. The round-1 DA procedure is as follows:

Step	s_1	s_2	s_3	s_4
1	$\vdots$	$\boxed{i_1}$	$\boxed{i_5}$	$\boxed{i_6}, i_3$
2		$\vdots$	$\boxed{i_5}, i_3$	$\vdots$
3	$\boxed{i_3}$		$\vdots$	
4	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_5}$	$\boxed{i_6}$

As all schools become essentially underdemanded, we can stop. The matching produced by the algorithm is constrained efficient, and i_1 and i_3 benefit from trading. Although i_3 and i_5 can still benefit from trading with each other, matching i_5 with s_1 will violate the priority of i_2. In fact, i_2 only yielded her priority at s_1 to i_3–who had the same priority as her before tie-breaking–but not to i_5.

5 Conclusion

We revisit the school choice problem with consent studied by Kesten, and propose a new perspective of focusing on unimprovable students for the design of algorithms. This allows us to design algorithms that simplify existing algorithms, for both school choice problems with consent and school choice problems with weak priorities. For the former, the simplification is on the transparency of consenting incentives and computational complexity, while for the latter, the simplification is on computational complexity and tractability. Since the efficiency losses associated with both fixed tie-breaking and stability are potentially significant in practice, our simplification has practical importance.[8]

A Appendix

A.1 Proof of Lemma 1

We show that if matching μ Pareto dominates a non-wasteful matching ν and student i's assignment $\nu(i)$ is essentially underdemanded at ν, then $\mu(i) = \nu(i)$. Suppose, instead, that $\mu(i) P_i \nu(i)$. Since matching ν is non-wasteful and student i desires school $\mu(i)$ at

[8] Abdulkadiroğlu, Pathak and Roth (2009) empirically document the extents of both types of efficiency losses in NYC high school match.

matching ν, $\mu(i)$ must be fully assigned at matching ν. Let $F^{\nu} \subset S$ be the set of schools excluding $\nu(i)$ that are fully assigned at matching ν. Then, $\mu(i) \in F^{\nu}$. Since at matching μ, student i obtains a seat from schools in F^{ν}, there must be at least one student, denoted by student j, who is matched with some school in F^{ν} at ν but is matched with some school $\mu(j) \notin F^{\nu}$ at matching μ. By assumption, $\mu(j)P_j\nu(j)$. Due to the non-wastefulness of ν and because j desires $\mu(j)$ at ν, $\mu(j)$ cannot have unfilled seats at ν either. Therefore, the only possibility is that $\mu(j) = \nu(i)$. Since $\nu(i) \notin F^{\nu}$, it must be the case that $\nu(i)P_j\nu(j)$. This implies that $\nu(i)$ cannot be (tier-0) underdemanded at ν. By assumption, for all students matched at underdemanded schools at ν, $\mu(i) = \nu(i)$.

Now suppose that $\nu(i)$ is tier-1 underdemanded at matching ν. Then $\nu(i)P_j\nu(j)$ implies that student j must be tier-0 underdemanded at matching ν, thus $\mu(j) = \nu(j)$. This is not possible, since $\mu(j) = \nu(i) \neq \nu(j)$. Consequently, $\nu(i)$ is not tier-1 underdemanded at ν, and the assignments of all students matched with tier-1 underdemanded schools do not change from matching ν to matching μ. By induction, $\nu(i)$ is not tier-k underdemanded at ν for any integer $k \geq 0$. This contradicts with the assumption that $\nu(i)$ is essentially underdemanded at matching ν.

A.2 Proof of Lemma 2

For each $k \geq 0$, let $(P^k, \succ^k)$ denote the school choice problem of round-k, which consists of only the schools and students that remain in round-k after removal, and let matching α_k denote the matching produced by the k-th round of the algorithm, in which the students who remain after removal in round-k are matched by round-k DA, and students removed in or before round-k are matched with the seats they are removed together with. Let α denote the eventual matching produced by the algorithm. Also, for each $k \geq 0$, let $UD_k \subset I$ denote the set of students matched with underdemanded schools at the round-k

DA matching $DA(P^k, \succ^k)$.

By definition of the algorithm, for every student removed before or at the beginning of round-k, her assignment is the same at α_k and α_{k-1}. For every student i who remains after removal in round-k, $\alpha_k(i) = DA(P^k, \succ^k)(i)$ and $\alpha_{k-1}(i) = DA(P^{k-1}, \succ^{k-1})(i)$. Note that for these students, their matches at $DA(P^{k-1}, \succ^{k-1})$ consist a stable matching at the round-k problem $(P^k, \succ^k)$, due to the stability of $DA(P^{k-1}, \succ^{k-1})$ at the larger problem $(P^{k-1}, \succ^{k-1})$. Since for the problem $(P^k, \succ^k)$, DA selects the student-optimal stable matching that weakly Pareto dominates all stable matchings of $(P^k, \succ^k)$ under P^k (Gale and Shapley, 1962), we know that in particular, it weakly Pareto dominates the restriction of $DA(P^{k-1}, \succ^{k-1})$ under P^k. Due to the specific way that P^k is modified from P, $DA(P^k, \succ^k)$ also weakly Pareto dominates the restriction of $DA(P^{k-1}, \succ^{k-1})$ under P.

This lemma can also be derived from rank monotonicity (Chen, 2013), or, more generally, weak Maskin monotonicity (Kojima and Manea, 2010); both are axioms used to characterize DA.

A.3 Proof of Proposition 1

It is sufficient to show that at each round of the algorithm, an underdemanded school exists at the DA outcome of that round. First, the null school is underdemanded at the round-0 outcome $DA(P, \succ)$. Now consider round-1. Due to arguments in the proof of Lemma 2, for students who remain after removal in round-1, $DA(P^1, \succ^1)$ weakly Pareto dominates $DA(P, \succ)$. Therefore, none of them is unmatched in round-1. Suppose student i is matched at the last step of round-1 DA and $DA(P^1, \succ^1)(i) = s$. Then s must be underdemanded at $DA(P^1, \succ^1)$. Otherwise, when i applies to s, admission at s must have been full, and acceptance of i must crowd out another student from s, who will be matched at a later step than i. We have a contradiction. The existence of an underdemanded school in later rounds follows from the same argument.

A.4 Proof of Theorem 1

Note that the result for Pareto efficiency can be viewed as a special case of constrained efficiency, where there are zero non-consenting students. As a result, we only need to show that when there are students who do not consent, the simplified EADAM is constrained efficient. We divide the proof into two parts. Some of the notations used below are defined in the proof of Lemma 2.

Part I. We begin by showing that the eventual matching α produced by the simplified EADAM does not violate the priority of any non-consenting student. Suppose student i does not consent and is matched with an underdemanded school in round-k DA. For any student removed before or together with i, her assignment is not desired by i–because if i desires her assignment, her assignment would not have been underdemanded. Therefore, such student cannot violate i's priority. For every student who remains in the problem after i is removed, if matching her with any school violates i's priority, then that school is placed below $\emptyset$ in her preference at the beginning of round-$(k+1)$ by the algorithm. Therefore, such students will not violate i's priority either.

Part II. We now show that any matching that Pareto dominates α must violate the priority of some non-consenting student. We already know from Lemma 1 that if μ Pareto dominates the round-0 outcome α_0 and does not violate the priority of any non-consenting student (under the original preference profile P), then for all $i \in UD_0, \mu(i) = \alpha_0(i)$.

Lemma 4. *Let round-$k, k \geq 1$, be an arbitrary round of the simplified EADAM. Suppose for each*

$0 \leq l \leq k-1$, if μ Pareto dominates α_l and does not violate the priority of any non-consenting student (under the original preference profile P), then $i \in UD_l$ implies $\mu(i) = \alpha_l(i)$. Then if μ Pareto dominates α_k and does not violate the priority of any non-consenting student (under the original preference profile P), then $i \in UD_k$ implies $\mu(i) = \alpha_k(i)$.

Proof. Suppose for some $i \in UD_k, \mu(i) P_i \alpha_k(i)$. Since μ Pareto dominates α_k under P, μ also Pareto dominates α_l for each $0 \leq l \leq k-1$ due to Lemma 2. Since μ does not violate the priority of any non-consenting student, by assumption of the lemma, $\mu(i) = \alpha_l(i)$, for each $i \in UD_l$ and each $0 \leq l \leq k-1$. Therefore, μ Pareto dominates α_k implies that for students who remain after removal in round-k, μ Pareto dominates $DA(P^k, \succ^k)$ under P. So either μ also Pareto dominates $DA(P^k, \succ^k)$ under P^k or for some i' who remains in round-k, $\mu(i')$ is placed below $\emptyset$ in $P^k_{i'}$ before round-k. In the former case, by applying Lemma 1 on $(P^k, \succ^k)$, we know that for all $i \in UD_k, \mu(i) = DA(P^k, \succ^k)(i) = \alpha_k(i)$. This contradicts with the assumption that $\mu(i) P_i \alpha_k(i)$ for some $i \in UD_k$. In the latter case, μ matches some student who remains after removal in round-k with a school placed below $\emptyset$ in earlier modifications of her preference. Again by definition of the algorithm, μ violates the priority of some non-consenting student removed before or in round-k. We have a contradiction.

$\square$

Since the case of $k = 1$ holds, by induction (due to Lemma 4), for each $k \geq 1$, if μ Pareto dominates α_k and does not violate the priority of any non-consenting student (under the original preference profile P), then $i \in UD_k$ implies $\mu(i) = \alpha_k(i)$.

We now prove the theorem. Let round-K be the last round of the simplified EADAM algorithm, which produces the eventual outcome $\alpha = \alpha_K$. Since μ Pareto dominates α, it Pareto dominates $\alpha_0, \ldots, \alpha_K$. By assumption, μ does not violate the priority of any non-consenting student. Due to Lemma 4, for each $0 \leq k \leq K$, if $i \in UD_k$, then $\mu(i) = \alpha_k(i)$. That is, at matching μ, each student $i \in I$ is matched with the seat that she is removed together with. Therefore, $\mu = \alpha$. We have a contradiction.

A.5 Proof of Lemma 3

We prove a more general result. Consider any school choice problem $(P, \succ)$ with consent. Let step t be the last step of the DA procedure in which interrupters are rejected, and let step $\hat{t}$ be the last step of the DA procedure in which consenting interrupters are rejected.

Lemma 5. *If school $DA(P, \succ)(i)$ accepts new students after step $\hat{t}$, then student i is unimprovable at $DA(P, \succ)$ without violating the priorities of non-consenting students.*

We divide the proof into two progressive parts.

Part I. We want to show that if school s accepts new students after step t, then s is essentially underdemanded at $DA(P, \succ)$. Suppose step $t' > t$ is the last step at which s accepts new students. Then s must have not rejected any student before step t'. This is because otherwise, when s accepts new students at step t', due to capacity constraints, it needs to reject tentatively accepted students, and we also know that during the tentative acceptance of these rejected students, other students must have been rejected by s. That is, at the same step at which s accepts these new students, it must be rejecting interrupters. This is in contradiction with our assumption that step t is the last step at which interrupters are rejected.

If s never rejects students throughout the DA procedure, then it is underdemanded and therefore essentially underdemanded. Otherwise, s must have rejected some student at step $t'' > t'$. When this rejected student applies to her assignment at the DA matching, similar to our argument on s, her assignment must have not rejected any student. If the assignments of all students rejected by s are underdemanded, then s is tier-1 underdemanded and therefore essentially underdemanded. If not, we will consider the students rejected by the assignments of these rejected students, and so on. Since the DA procedure stops within finitely many steps, by induction, we can show that the assignment of every one of these rejected students is underdemanded of some tier and therefore is essentially underdemanded.

Part II. We want to show that if school $\hat{s}$ accepts new students after step $\hat{t}$, then $DA(P, \succ)(i) = \hat{s}$ implies that student i is not Pareto improvable at $DA(P, \succ)$. Suppose no non-consenting interrupter is rejected after step $\hat{t}$. Then step $\hat{t}$ will be the last step in which interrupters are rejected. Due to Part I, school $\hat{s}$ must be essentially underdemanded, and therefore the result holds.

Otherwise, there are interrupters who are rejected after step $\hat{t}$ and none of them consents. Let's begin with the lastly rejected interrupters. We already know that they are unimprovable, and by assumption, they don't consent. For each one of these interrupters–for instance, student i–consider the last school s' that she interrupts. If any other student desires s', then the student is either rejected by s' earlier than i is rejected, in which case she cannot be improved to s'–since that will violate i's priority at s'–or rejected by s' not earlier than i, in which case she must be matched at an essentially underdemanded school and, again, is not improvable. Since no student can be Pareto improved to s' (without violating non-consenting students' priorities), no student matched at s' can be Pareto improved.

Define the updated set of underdemanded schools $\bar{S}_{ud}$ as the union of the set of lastly interrupted schools and the set of essentially underdemanded schools. If we further view $\bar{S}_{ud}$ as the updated set of tier-0 essentially underdemanded schools, then we can define the updated set of essentially underdemanded schools $\bar{S}_{eud}$ in the same way as in Definition 2. From the above, students matched in $\bar{S}_{eud}$ cannot be improved without violating non-consenting students' priorities.

Now consider the second-to-last step after step $\hat{t}$, in which interrupters are rejected. Due to the same arguments used to prove Part I, if school $s'' \notin \bar{S}_{ud}$ accepts new students after this step, then $s'' \in \bar{S}_{eud}$. Therefore, students matched at s'' will be unimprovable. As the interrupters rejected in this second-to-last step must all be matched with schools like s'', their assignments are also unimprovable. Consequently, students matched with the schools that they interrupted are also unimprovable. We can now construct a larger updated set of underdemanded schools by taking the union of $\bar{S}_{eud}$ and the interrupted school at this second-to-last step, and the rest of the proof follows from induction.

A.6 Proof of Theorem 3

For notational convenience, let's denote the simplified EADAM (with underdemanded schools) as φ_{ud} and denote Kesten's EADAM as φ_{Kesten}. We need to show that for any fixed school choice problem $(P, \succ)$, $\varphi_{ud}(P, \succ) = \varphi_{Kesten}(P, \succ)$. As defined before, for each $k \geq 0, \alpha_k$ denotes the matching produced by the round-k DA of φ_{ud} and $UD_k \subset I$ denotes the set of students who are matched with underdemanded schools at α_k. To begin, since both φ_{ud} and φ_{Kesten} weakly Pareto dominates DA, due to Lemma 1, $\varphi_{ud}(P, \succ)(j) = \varphi_{Kesten}(P, \succ)(j) = DA(P, \succ)(j), \forall j \in UD_0$.

We now show by induction that for each $k \geq 1$, if for all $j \in UD_0 \cup UD_1 \cup \cdots \cup UD_{k-1}, \varphi_{ud}(P, \succ)(j) = \varphi_{Kesten}(P, \succ)(j)$, then for all $j' \in UD_k$, $\varphi_{ud}(P, \succ)(j') = \varphi_{Kesten}(P, \succ)(j')$. Again due to Lemma 1 and because students in UD_k are matched with underdemanded schools at α_k, it is sufficient to show that $\varphi_{Kesten}(P, \succ)$ weakly Pareto dominates α_k and does not violate the priorities of any non-consenting students removed before round-k DA in φ_{ud}. The latter is satisfied, because the algorithm φ_{Kesten} does not remove applications of non-consenting students'.

Consider the k-th round of φ_{ud}. Suppose, instead, there are students who remain in the problem and strictly prefer α_k to $\varphi_{Kesten}(P, \succ)$. Let the set of such students be denoted by $I_{\alpha_k \succ \varphi}$. Let round r be the first round of φ_{Kesten} such that at the DA matching it produces, the assignment of some $i \in I_{\alpha_k \succ \varphi}$ becomes not Pareto improvable (without violating the priorities of non-consenting students). This construction is inspired by Bando (2014) (proof of Claim 7, Part [i]). Due to Lemma 3, when the application of some consenting interrupter is removed in some round of φ_{Kesten}, her assignment is already unimprovable. Put together, no consenting interrupter i' whose applications are removed before round r of φ_{Kesten} can be in $I_{\alpha_k \succ \varphi}$; otherwise, given that the round in which i' becomes an interrupter and thus unimprovable is earlier than round r, round r will not be the first round of φ_{Kesten} that satisfies the condition above.

We have constructed a student i who remains in the round-k DA of φ_{ud} such that $\alpha_k(i) P_i \varphi_{Kesten}(P, \succ)(i)$. Moreover, during φ_{Kesten}, if applications of any consenting interrupter i' have been removed before round r–the round in which i becomes unimprovable–then $\varphi_{Kesten}(P, \succ)(i') R_{i'} \alpha_k(i')$. Now consider a preference profile of students in which the set of applications removed is the union of the set of applications (of consenting students) removed before round-k DA during φ_{ud} and the set of applications (of consenting students) removed before round-r DA during φ_{Kesten}. Let μ be the matching produced by DA after removing these applications from $(P, \succ)$.

Then $\mu(i) = \varphi_{Kesten}(P, \succ)(i)$. This is because i already becomes Pareto unimprovable at the round-r DA matching of φ_{Kesten} and the additional removal of applications from consenting students in $UD_0 \cup UD_1 \cup \cdots \cup UD_{k-1}$ does not hurt any student at the round-r DA matching of φ_{Kesten}. Therefore, the assignment of i will not be affected. Likewise, $\mu(i) R_i \alpha_k(i)$. This is because given the applications removed before the round-k

DA of φ_{ud}, the additional applications removed are all from consenting interrupters who weakly prefer $\varphi_{Kesten}(P, \succ)$ to α_k. As a result, the additional applications removed from any consenting interrupter i' must be strictly better than $\alpha_k(i')$. Due to exactly the same argument behind the proof of Lemma 2, the additional removal can only weakly improve the assignments of the remaining students (including student i) on α_k.

Since $\mu(i) = \varphi_{Kesten}(P, \succ)(i)$ and $\mu(i) R_i \alpha_k(i)$, we have $\varphi_{Kesten}(P, \succ)(i) R_i \alpha_k(i)$, which is in contradiction with the assumption that $\alpha_k(i) P_i \varphi_{Kesten}(P, \succ)(i)$. Therefore, we have proven that for all $j \in UD_k$, $\varphi_{ud}(P, \succ)(j) = \varphi_{Kesten}(P, \succ)(j)$. By induction on k, we have $\varphi_{Kesten}(P, \succ) = \varphi_{ud}(P, \succ)$.

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