Hierarchical Exchange Rules and the Core in Indivisible Objects Allocation

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Abstract

We study the allocation of indivisible objects under the general endowment structures proposed by Pápai (2000)–the consistent inheritance structures–which specify the initial endowment of objects and also the inheritance of remaining objects after subsets of agents are matched and removed. For any consistent inheritance structure and any given matching, we define the contingent endowment of an agent as the maximal set of objects that she can feasibly inherit given the consistency of endowments. Based on contingent endowment, the concepts of individual rationality and core are then generalized. We show that for each consistent inheritance structure, Pápai's hierarchical exchange rule produces the unique core allocation and is characterized by individual rationality, Pareto efficiency, and strategy-proofness.

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1 Introduction

The allocation of heterogenous indivisible objects (called houses) to agents, where each agent demands at most one object and no monetary transfers are allowed, is common in real-life economic activities.[1] A well-designed allocation mechanism should allocate the objects efficiently among agents, and should not create incentives for agents to manipulate their preferences. Pápai (2000) proposes the class of hierarchical exchange rules which includes many existing mechanisms–the serial dictatorship mechanism (Svensson, 1994) and Gale's top trading cycles mechanism (TTC; Shapley and Scarf, 1974) in particular–as special cases.[2]

Joining the efforts of Svensson and Larsson (2005), we provide a unified understanding of hierarchical exchange rules by extending our understanding on Gale's TTC and the Shapley-Scarf housing market.[3] Hierarchical exchange rules are more general than Gale's TTC crucially because they are defined for more general endowment structures. In a Shapley-Scarf housing market, each agent is endowed with exactly one house and is available for trading in the market. More generally, each hierarchical exchange rule is associated with a consistent inheritance structure, which specifies how objects are ini-

[1] Examples include the assignment of school seats or dormitory rooms to students (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu and Sönmez, 1999), the allocation of kidneys for transplantation (Roth et. al., 2004), the general problem of assigning individuals to positions (Hylland and Zeckhauser, 1979), and so on.

[2] Pápai (2000) characterizes the class of hierarchical exchange rules with group strategy-proofness, Pareto efficiency, and reallocation-proofness.

[3] It is well-known that for the Shapley-Scarf housing market, TTC selects the unique core of this market (Shapley and Scarf, 1974; Roth and Postlewaite, 1977) and is characterized by individual rationality, strategy-proofness, and Pareto efficiency (Ma, 1994).

tially endowed to agents and under each occasion of a subset of objects being assigned and removed, specifies the endowment of the remaining objects. After specification of endowments, the rule implements trades in top-trading cycles, just as that of TTC.

We begin with a careful examination of the consistency condition of the inheritance structures. By consistency, once an agent is endowed with an object, she keeps owning it until she is matched and removed. For this to hold, whenever any agent is removed, her assignment must not be from the endowments of the remaining agents. Consequently, consistency restricts the submatchings that can be feasibly removed and hence the markets that can feasibly appear during the inheritance procedure. Starting with the null submatching, we recursively define feasible submatchings as the minimal enlargements of previously constructed feasible submatchings that satisfy the consistency criterion. Interestingly, each of such minimal enlargement corresponds exactly to a trading cycle among agents with nonempty endowments.

From a general perspective, we can view every hierarchical exchange market as a market consisting of agents with contingent endowments. The contingent endowment of each agent, which is endogenously determined by the inheritance structure and the assignments of the other agents, is defined as follows. For any given matching and agent, we show that there exists a maximal feasible submatching that excludes the agent. And due to consistency, the endowments of each agent weakly enlarges as more of the other agents are feasibly matched and removed. Based on these, we define the contingent endowment of an agent as her endowment specified by the inheritance structure after the maximal feasible matching that excludes her is removed.

With the concept of contingent endowment being properly defined for the agents, we can extend the analysis of Shapley-Scarf markets to hierarchical exchange markets. Fix an inheritance structure. We say that an agent is individually rational at a matching if her assignment is weakly preferred to all objects in her contingent endowment at this matching. Similarly, a matching is said to be in the core if there do not exist any coalition and any alternative matching, such that at the alternative matching, agents in the coalition are Pareto improved and are not matched with objects contingently endowed to the outsiders. These extensions provide us with a general perspective on hierarchical exchange markets and consequently, allow us to extend the classical results of Shapley and Scarf (1974), Roth and Postlewaite (1977), and Ma (1994) from Shapley-Scarf housing market to hierarchical exchange markets.

Our contributions are of two folds: (i) by properly defining the concept of contingent endowment, we unify the understanding of hierarchical exchange markets with that of Shapley-Scarf housing market from an ex ante (or static) perspective; and (ii) by examining the implication of consistency, we provide a foundation for the connection between feasible submatchings and trading cycles. Svensson and Larsson (2005) are the first to extend the concepts of individual rationality and core, and the characterizations to hierarchical exchange markets. The approach that they take is from an interim perpective. They define feasible submatchings by directly referring to trading cycles, and define individual rationality and core by explicitly referring to tradings in interim stages of the hierarchical exchanges.

Our work also relates to other papers in the literature. Abdulkadiroğlu and Sönmez (2003) extend the TTC mechanism to priority-based allocation problems, and Abdulkadiroğlu et. al. (2013) characterize this mechanism with a set of intuitive axioms. By interpreting a priority structure as an inheritance structure, our work provides an alternative characterization of this mechanism. For allocation problems, Pycia and Ünver (2015) characterize the class of group strategy-proof and Pareto efficient mechanisms, which includes hierarchical exchange rules as special cases. They reformulate inheritance structures, which are originally formulated by inheritance trees in Pápai (2000), as a collection of mappings defined on the set of submatchings. Throughout, we closely follow their notations. By specifying interim endowments as either ex ante endowment or ex post assignment, Ekici (2013) successfully uses a static version of individual rationality to capture that of interim stages in the hierarchical exchange. In spirit, our approach is very closely related to his. Sönmez (1999) generalizes the relationship between core and strategy-proofness to a wide range of allocation problems with fixed endowment structures.

This paper is organized as follows. In Section 2, basic notations are defined. In Section 3, we examine the consistency of inheritance structures and define hierarchical exchange rules. In Section 4, we present the extensions of characterizations. Section 5 concludes. All proofs are relegated into Appendix A.

2 Notations

We study the house allocation problem in which a finite set of heterogeneous houses (indivisible objects) are to be allocated to a finite set of agents, and no monetary transfer is allowed. Let the set of houses be $H = \{h_1, \ldots, h_m\}$ and the set of agents be $I = \{i_1, \ldots, i_n\}$. Each agent demands at most one house. For each agent $i \in I$, let P_i denote a complete, irreflexive, and transitive preference relation over $H \cup \{\emptyset\}$, where $\emptyset$ is the null house which has unlimited copies, and matching with $\emptyset$ represents being unmatched; let R_i denote the weak preference relation associated with P_i. Also, let $P \equiv (P_i)_{i \in I}$ denote a preference profile, and let $\mathbf{P}$ denote the set of all preference profiles. For each $J \subset I$, denote $(P_j)_{j \in J}$ by P_J, and as usual, $-J \equiv I \backslash J$.

A **house allocation problem** is defined as a triple (I, H, P), or, simply, as P. A **match-**

ing (or, an allocation) is a function $\mu : I \to H \cup \{\emptyset\}$ such that for $i \neq i', \mu(i) = \mu(i')$ only if $\mu(i) = \emptyset$. Let $\mathcal{M}$ denote the set of all matchings. An allocation mechanism is a mapping $\varphi : \mathbf{P} \to \mathcal{M}$ that selects a matching for each preference profile. Let $\varphi_i(P_i, P_{-i})$ denote the assignment of agent i under φ at the preference profile (P_i, P_{-i}).

A matching ν Pareto dominates matching μ w.r.t. P if for all $i \in I, \nu(i) R_i \mu(i)$, and for some $j \in I, \nu(j) P_j \mu(j)$. A matching μ is Pareto efficient w.r.t. P if it is not Pareto dominated by any matching. A mechanism φ is Pareto efficient if it selects a Pareto efficient matching for every preference profile. A mechanism φ is strategy-proof if it is a weakly dominant strategy for each agent to report her preference truthfully. Formally, φ is strategy-proof if for all agent $i \in I$, all preference profile $(P_i, P_{-i}) \in \mathbf{P}$ and all P'_i, $\varphi_i(P_i, P_{-i}) R_i \varphi_i(P'_i, P_{-i})$. Likewise, a mechanism φ is group strategy-proof if there do not exist $J \subset I, P$ and $P'_J = (P'_i)_{i \in J}$ such that for all $i \in J, \varphi_i(P'_J, P_{-J}) R_i \varphi_i(P)$, and for some $j \in J, \varphi_j(P'_J, P_{-J}) P_j \varphi_j(P)$.

3 Hierarchical exchange rules

We study the hierarchical exchange rules proposed by Pápai (2000), which generalize Gale's top trading cycles mechanism (TTC, Shapley and Scarf, 1974) from housing markets–where each agent is endowed with a single unit of house–to general endowment structures called inheritance structures. In Pápai (2000), an inheritance structure is defined in the form of inheritance trees which specifies how each house is inherited to another agent after its owner is matched and removed. Inheritance structures are later reformulated by Pycia and Ünver (2015) as a collection of inheritance mappings which specifies the inheritance of the remaining houses, after the removal of each submatching. In below, we follow the formulation of the latter more closely.

3.1 Consistent inheritance structures

Before we proceed, more notations need to be defined. A submatching is an one-to-one function $\sigma : J \rightarrow H \cup \{\emptyset\}$, where $J \subset I$. By slightly abusing notation, we also use $\emptyset$ to denote the empty submatching in which no agent is matched. Let the set of all submatchings be $\mathcal{S}$, and the set of all strict submatchings be $\bar{\mathcal{M}} = \mathcal{S} - \mathcal{M}$.

For any submatching σ, let $I_\sigma \subset I$ denote the set of agents and $H_\sigma \subset H$ denote the set of houses matched in σ. For simplicity, we also write $i \in \sigma$ if $i \in I_\sigma$ and $h \in \sigma$ if $h \in H_\sigma$. Let $\bar{I}_\sigma = I - I_\sigma$ denote the set of remaining agents and $\bar{H}_\sigma = H - H_\sigma$ the set of remaining houses after the submatching σ has been removed from the market. Given two submatchings σ, σ', we say that σ is a submatching of σ', denoted by $\sigma \subset \sigma'$, if $I_\sigma \subset I_{\sigma'}$ and for every $i \in I_\sigma, \sigma(i) = \sigma'(i)$. For any pair of submatchings σ and σ' such that $\sigma(i) = \sigma'(i)$ for all $i \in I_\sigma \cap I_{\sigma'}$, let $\sigma \cap \sigma'$ and $\sigma \cup \sigma'$ be submatchings that coincide with σ and σ' at common agents, with $I_{\sigma \cap \sigma'} = I_\sigma \cap I_{\sigma'}$ and $I_{\sigma \cup \sigma'} = I_\sigma \cup I_{\sigma'}$.

Definition 1 (Pycia and Ünver, 2015)**.** *An **inheritance structure** of the houses is a collection of mappings $\{c_\sigma : \bar{H}_\sigma \rightarrow \bar{I}_\sigma\}_{\sigma \in \bar{\mathcal{M}}}$. For simplicity, denote the inheritance structure by $\{c_\sigma\}$.*[4]

For each strict submatching σ, c_σ is a one-to-one mapping which specifies how the remaining houses are endowed to the remaining agents after the removal of σ, or equivalently, specifies how the remaining agents inherit the remaining houses. In particular, $c_\emptyset : H \rightarrow I$ specifies the initial endowments. For any submatching σ and $i \notin I_\sigma$, $c_\sigma^{-1}(i)$ is the set of houses endowed to agent i after σ is removed. Likewise, for any $J \subset \bar{I}_\sigma$, $c_\sigma^{-1}(J) = \cup_{j \in J} c_\sigma^{-1}(j)$ is the set of all houses endowed to agents in J after the removal of σ.

Example 1 (Priority structures)**.** Priority structures are simple examples of inheritance structures. For each house $h \in H$, let $\succ_h$ denote a (strict) priority ordering of agents

[4]The inheritance structure specifies inheritance at all strict submatching $\sigma \in \bar{\mathcal{M}}$. This corresponds to conditions (C.1)-(C.3) in Pápai (2000) and "closed" endowment rule in Svensson and Larsson (2005).

which specifies the ordering of inheritance for house h. A priority structure is a priority profile $(\succ_h)_{h\in H}$. Then the associated inheritance structure can be defined as follows. For any submatching σ, let $c_\sigma : \bar{H}_\sigma \to \bar{I}_\sigma$ be a mapping such that for each $h \in \bar{H}_\sigma, c_\sigma(h) = \arg\max_{\succ_h}\{i \in \bar{I}_\sigma\}$. That is, whenever a submatching σ is removed, each remaining house $h \in \bar{H}_\sigma$ is endowed to the agent with highest priority among the remaining agents according to $\succ_h$.

We are interested in inheritance structures that are consistent in the sense that whenever an agent is endowed with a house after some submatching is removed, she keeps owning until she is matched and removed. We use the example below to illustrate the concept of feasible submatchings, which is crucial for the formal definition of consistency.

Example 2. Consider the following priority structure. Suppose $I = \{1, 2, 3\}, H = \{a, b, c\}$. The priorities at houses and the corresponding initial endowment are described by the left and right table, respectively:

$\succ_a$	$\succ_b$	$\succ_c$
1	1	2
2	3	3
3	2	1

$c_\emptyset^{-1}(1)$	$c_\emptyset^{-1}(2)$	$c_\emptyset^{-1}(3)$
$\{a, b\}$	$\{c\}$	$\emptyset$

At the initial endowment, the submatching $\sigma = \{(2, a)\}$ is not "feasible", because after the removal of this submatching, agent 1 still remains, but she will not own a anymore, no matter how the remaining houses are endowed.

Likewise, the submatching $\sigma' = \{(1, b), (3, a)\}$ is also not "feasible". This is because after 1 is "feasibly" matched and removed with b, the priority structure endows a to 2. Therefore, if 3 leaves with a, then the consistency of 2's endowment is violated.

We say that a submatching σ is minimal in a set $\mathcal{A}$ of submatchings, if $\sigma \in \mathcal{A}$ and there does not exist $\sigma' \in \mathcal{A}$ such that $\sigma' \subsetneq \sigma$.

Definition 2. *The set of* ***feasible*** *submatchings associated with an inheritance structure* $\{c_\sigma\}$*, denoted by* $\mathcal{F}_c$*, is a subset of* $\mathcal{S}$ *such that:*

(i) $\emptyset \in \mathcal{F}_c$*;*

(ii) $\sigma \in \mathcal{F}_c$ *if there exists* $\sigma' \in \mathcal{F}_c$ *with* $\sigma' \subsetneq \sigma$ *such that* σ *is minimal in* $\{\tilde{\sigma} : \tilde{\sigma} \supsetneq \sigma'$ *and* $H_{\tilde{\sigma}} - H_{\sigma'} \subset c_{\sigma'}^{-1}(I_{\tilde{\sigma}} - I_{\sigma'})\}$.

By definition, the null submatching $\emptyset$ is feasible, and any submatching is feasible only if it is a minimal enlargement of a feasible submatching that satisfies a consistency criterion. To be precise, suppose we are at the initial market with endowments specified by $c_\emptyset$. For any submatching $\sigma \supset \emptyset$ to be matched and removed without violating the consistency of the remaining agents' endowments, any agent in σ can only be matched with houses owned by agents in σ. That is, $H_\sigma - H_\emptyset \subset c_\emptyset^{-1}(I_\sigma - I_\emptyset)$. Furthermore, σ not only needs to satisfy this consistency criterion with respect to $\emptyset$, but also with respect to every strict "feasible" submatching $\sigma' \subset \sigma$, so that from the perspective of the market specified by $c_{\sigma'}$ (after the removal of σ'), σ is also well-justified. For this reason, the minimality condition is imposed in the construction of feasible submatchings.

For any inheritance structure $\{c_\sigma\}$, starting with the empty submatching, the set of feasibly submatchings $\mathcal{F}_c$ can be inductively constructed. In fact, after any feasible submatching σ is removed, each minimal enlargement of σ corresponds exactly to a trading cycle among the remaining agents, at endowments specified by c_σ.[5] That is, in each level of the hierarchical market, only trading cycles are feasible.

[5] At the market obtained after the removal of a feasible submatching σ, where agents' endowments are specified by $c_\sigma : \bar{H}_\sigma \to \bar{I}_\sigma$, a tradinig cycle consists a subset of agents $i_1, i_2, \ldots, i_k = i_1$, such that for each $1 \leq l \leq k-1$, i_l points to an object in $c_\sigma^{-1}(i_{l+1})$.

We now modify the definition of consistency provided in Pycia and Ünver (2015) by incorporating feasibility. The set of feasible submatchings is the maximal domain for the consistency of endowments.

Definition 3. *An inheritance structure $\{c_\sigma\}$ is **consistent** if for any $\sigma, \sigma' \in \mathcal{F}_c, \sigma \subset \sigma'$, and $i \in \bar{I}_{\sigma'}$,*

$$c_\sigma^{-1}(i) \subset c_{\sigma'}^{-1}(i).$$

That is, an inheritance structure is consistent if after any feasibly matching σ is removed (hence agent i is endowed with the set of objects $c_\sigma^{-1}(i)$), when more of other agents are feasibly matched and the removed matching enlarges to σ', i's endowment weakly enlarges to $c_{\sigma'}^{-1}(i)$. Consistency corresponds to Pápai (2000)'s assurance rule for inheritance tree and Svensson and Larsson (2005)'s monotonicity in endowment rule. It is straightforward that every priority structure is a consistent inheritance structure. And for each consistent inheritance structure, the collection of mappings defined on feasible submatchings, $\{c_\sigma\}_{\sigma \in \mathcal{F}_c}$, is equivalent to a canonical form of inheritance trees. To specify a consistent inheritance structure, we only need to inductively specify the endowments at feasible submatchings.

This notion of feasibility we propose here coincides with that defined by Svensson and Larsson (2005). The constructions above show that the connection between feasibility and trading cycles is a direct implication of the consistency of the inheritance structure. Therefore, these constructions can be viewed as a foundation of the definition of feasibility.

The result below illustrates the structure of the set of feasible submatchings.

Proposition 1. *Given a consistent inheritance structure $\{c_\sigma\}$, for all $\sigma, \sigma' \in \mathcal{F}_c$, we have*

(i) *if $\sigma \subset \sigma'$, then $H_{\sigma'} - H_\sigma \subset c_\sigma^{-1}(I_{\sigma'} - I_\sigma)$;*

(ii) *if σ, σ' are both submatchings of a matching, then $\sigma \cap \sigma', \sigma \cup \sigma' \in \mathcal{F}_c$.*

Part (**i**) of the proposition states that any feasible submatching is indeed consistent with respect to all feasible submatchings of it, and part (**ii**) states that the union and intersection of feasible submatchings of any given matching are also feasible.

3.2 Hierarchical exchange rules

From now on, we restrict attention to consistent inheritance structures. An **allocation problem with inheritance** is a pair $(P, \{c_\sigma\})$. For any such problem, Pápai (2000) proposes the hierarchical exchange rule–an algorithm which operates as follows:[6]

Step 1. Houses are endowed to agents as specified by $c_\emptyset : H \to I$. Each agent with nonempty endowment points to the owner of her most favorite house. Due to the finiteness of the set of agents, there exist at least one cycle. For each agent that belongs to any of the cycles, match her with the house that she points to and remove her with her match. Denote the removed submatching by σ_1.

Step $t \geq 2$. After $\bar{\sigma}_{t-1} \equiv \sigma_1 \cup \cdots \cup \sigma_{t-1}$ is removed, the remaining houses are endowed to the remaining agents according to $c_{\bar{\sigma}_{t-1}} : \bar{H}_{\bar{\sigma}_{t-1}} \to \bar{I}_{\bar{\sigma}_{t-1}}$. Each remaining agent with nonempty endowment points to the owner of her most favorite remaining house. All top trading cycles are traded and removed as in Step 1. Denote the removed submatching of step by σ_t.

The algorithms stops when either all agents or all houses have been removed.

[6]Abdulkadiroğlu and Sönmez (2003) generalize Gale's TTC for priority structures with application to school choice. Their algorithm is exactly the same as Pápai (2000)'s hierarchical exchange rule, once we interpret priority structures as inheritance structures.

This algorithm stops within finitely many steps. Suppose it stops right after Step T, then the outcome of the hierarchical exchange rule is $\mu = \sigma_1 \cup \cdots \cup \sigma_T$.

4 Characterizations

Under inheritance structures, an agent may inherit houses which she does not previously own from other agents after those agents are matched and removed. Accordingly, the endowment of an agent is not determined ex ante, but is instead determined endogenously by the inheritance structure together with the assignments of others.

In below we first introduce the concept of contingent endowment, and based on that extend individual rationality and core to inheritance structures. Eventually, we extend the classical characterization theorems from Shapley-Scarf housing markets to hierarchical exchange markets. Our extensions capture the same intuition as that of Svensson and Larsson (2005); the major difference is in the formulations. In short, contingent endowment allows us to extend the concepts and characterizations from a static (or, ex ante) perspective, without explicitly referring to the interim trading procedure.

4.1 Extending Ma (1994)

Due to Proposition 1, given any matching μ and any agent i, there exists a maximal feasible submatching of μ that does not include i; let this submatching be denoted by $\sigma_{\max}(\mu \backslash i)$.

Definition 4. *For any consistent inheritance structure $\{c_\sigma\}$ and matching μ, the **contingent***

endowment *of agent i at μ is defined as*

$$\omega(i|\mu) = c^{-1}_{\sigma_{\max}(\mu\setminus i)}(i).$$

That is, $\omega(i|\mu)$ is the set of houses that i would have been endowed with at the contingency that the maximal submatching of μ that excludes i has been removed. It follows from consistency that $\omega(i|\mu)$ is the maximal set of endowment that agent i can feasibly inherit. Therefore, for any feasible submatching $\sigma \subset c_{\sigma_{\max}(\mu\setminus i)}, c^{-1}_{\sigma}(i) \subset \omega(i|\mu)$. Intuitively, after knowing the assignments of each other, each agent is able to recover her contingent endowment via recovering the maximal feasible submatching that excludes her. And as long as the agents agree on the consistency of the inheritance structure, they will agree on feasibility and hence on each other's contingent endowment.

Definition 5. *Agent i is **individually rational** at matching μ if*

$$\mu(i) R_i h, \forall h \in \omega(i|\mu).$$

The implication of individual rationality is of two folds: (**i**) the agent should at least obtain her most favorite from what she is endowed; and (**ii**) the agent should not leave if she can inherit better houses by staying longer.[7] For the Shapley-Scarf housing market, $\omega(i|\mu) = h_i$ for all μ, and hence the individual rationality reduces to $\mu(i) R_i h_i$. While by referring to contingent endowment, we define individual rationality in a compact way, the concept itself is equivalent to the corresponding definition of Svensson and Larsson (2005).

[7]In the same spirit, Kesten (2006) reinterprets the priority-based top trading cycles mechanism proposed by Abdulkadiroğlu and Sönmez (2003) as a hierarchical exchange rule with inheritance, and calls the property that "the allotment of an agent is no worse for him than an object he is assigned at some step" as individual rationality.

Given the extension of individual rationality that we propose, a unified understanding of hierarchical exchange rules can be obtained from the following characterization which extends Ma (1994)'s characterization of Gale's TTC for Shapley-Scarf housing markets.

Theorem 1. *For any house allocation problem with a consistent inheritance structure, an allocation mechanism is individually rational, Pareto efficient, and strategy-proof if and only if it is the hierarchical exchange rule.*

4.2 Core

In a Shapley-Scarf housing market, a matching μ is in the core if there do not exist any coalition $B \subset I$ and matching ν such that: (i) for any $i \in B$, $\nu(i)$ is owned by some agent in B, or equivalently, $\nu(i)$ is not owned by any agent in B^c; and (ii) for agents in B, matching ν Pareto dominates μ.

We extend the concept of core to allocation problems with inheritance structures.

Definition 6. *For any consistent inheritance structure, a matching μ is in the* ***core*** *if there do not exist any coalition $B \subset I$ and matching ν such that*

(i) $\nu(i) \notin \omega(j|\nu), \forall i \in B$ *and* $j \in B^c$;

(ii) $\nu(i) R_i \mu(i), \forall i \in B$, *and* $\nu(j) P_j \mu(j)$ *for some* $j \in B$.

Condition (**i**) requires that at ν, no agent in the coalition B is matched with houses in the contingent endowment of agents in B^c, and condition (**ii**) requires that ν Pareto dominates μ for agents in B.

We now extend the existence and uniqueness of core allocation results of Shapley and Scarf (1974) and Roth and Postlewaite (1977) from Shapley-Scarf market to allocation problems with inheritance.

Theorem 2. *For any house allocation problem with consistent inheritance structure, the hierarchical exchange rule selects the unique core allocation.*

5 Conclusion

Gale's top trading cycles mechanism, initially defined for Shapley-Scarf housing markets, is generalized to allow for inheritance in endowments by hierarchical exchange rules. Nevertheless, hierarchical exchange rules are defined by exactly the same top trading procedure. Therefore, it seems natural that we should try to understand these general rules from the perspective of Gale's TTC, which is well-studied.

To obtain a unified understanding of these mechanisms, we try to view hierarchical exchange market as a static market, in the same way as Shapley-Scarf housing market, where there is a set of agents, and each has her own endowment. We achieve this by generalizing the concept of endowment. After contingent endowment is defined, the rest of the generalization become straightforward. The definition of contingent endowment, in turn, crucially depends on agent's common understanding of the consistency of the inheritance structure and its consequence on the feasibility of matchings.

A Appendix

A.1 Proof of Proposition 1

Part (i). We need to show that if $\sigma, \sigma' \in \mathcal{F}_c$ and $\sigma \subset \sigma'$, then $H_{\sigma'} - H_\sigma \subset c_\sigma^{-1}(I_{\sigma'} - I_\sigma)$. Suppose instead $H_{\sigma'} - H_\sigma \nsubseteq c_\sigma^{-1}(I_{\sigma'} - I_\sigma)$. That is, there exists an agent $i \in I_{\sigma'} - I_\sigma$ such that $\sigma'(i) \in c_\sigma^{-1}(j)$, for some $j \notin I_{\sigma'}$. As a result, after σ' is removed, the house $\sigma'(i)$ is not owned by j any longer, i.e., $\sigma'(i) \notin c_{\sigma'}^{-1}(j)$. As a result, $\sigma \subset \sigma'$, $j \notin \sigma'$, but $c_\sigma^{-1}(j) \nsubseteq c_{\sigma'}^{-1}(j)$, a contradiction to consistency.

Part (ii). We need to show that if two feasible submatchings σ, σ' are both submatchings of a full matching μ, then both $\sigma \cup \sigma'$ and $\sigma \cap \sigma'$ are also feasible.

To show $\sigma \cap \sigma'$ is feasible, we first claim that for any feasible submatching $\tilde{\sigma} \subsetneq \sigma \cap \sigma'$, $H_{\sigma\cap\sigma'} - H_{\tilde{\sigma}} \subseteq c_{\tilde{\sigma}}^{-1}(I_{\sigma\cap\sigma'} - I_{\tilde{\sigma}})$. Note that the collection of all feasible submatchings of $\sigma \cap \sigma'$ is nonempty since $\emptyset$ is such an element. As a result, there exists a maximal feasible submatching in this collection in the sense that there is no other feasible submatching in-between this maximal submatching and $\sigma \cap \sigma'$. That is, $\sigma \cap \sigma'$ itself is feasible since it satisfies the "minimality" in Definition 2.

The above claim can be proved by contradiction. Suppose $\tilde{\sigma}$ is a feasible submatching of $\sigma \cap \sigma'$ but $H_{\sigma\cap\sigma'} - H_{\tilde{\sigma}} \nsubseteq c_{\tilde{\sigma}}^{-1}(I_{\sigma\cap\sigma'} - I_{\tilde{\sigma}})$. Then there exists $i \in I_{\sigma\cap\sigma'} - I_{\tilde{\sigma}}$ and $j \notin I_{\sigma\cap\sigma'} - I_{\tilde{\sigma}}$ such that $(\sigma \cap \sigma')(i) \in c_{\tilde{\sigma}}^{-1}(j)$. According to part (i), since $\sigma, \tilde{\sigma} \in \mathcal{F}_c$ and $\tilde{\sigma} \subset \sigma$, we have $(\sigma \cap \sigma')(i) \in H_\sigma - H_{\tilde{\sigma}} \subset c_{\tilde{\sigma}}^{-1}(I_\sigma - I_{\tilde{\sigma}})$. As a result, $j \in I_\sigma - I_{\tilde{\sigma}}$. Similarly, $\tilde{\sigma} \subset \sigma'$ also implies that this agent $j \in I_{\sigma'} - I_{\tilde{\sigma}}$. Therefore, $j \in I_{\sigma\cap\sigma'} - I_{\tilde{\sigma}}$, a contradiction.

We next show that $\sigma \cup \sigma'$ is feasible. Since $\sigma \cap \sigma'$ is feasible, by consistency, for any agent $j \in I_{\sigma'} - I_\sigma$, $c_{\sigma\cap\sigma'}^{-1}(j) \subset c_\sigma^{-1}(j)$. Moreover, since $\sigma \cap \sigma'$ and σ' are both feasible, there exists a sequence of feasible submatchings, denoted by, $\sigma \cap \sigma' = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k = \sigma'$

such that for any $\ell \leq k$, σ_ℓ is a minimal matching in $\{\tilde{\sigma} \supset \sigma_{\ell-1} : H_{\tilde{\sigma}} - H_{\sigma_{\ell-1}} \subset c^{-1}_{\sigma_{\ell-1}}(I_{\tilde{\sigma}} - I_{\sigma_{\ell-1}})\}$.

We next claim that $\sigma \cup \sigma_1$ is a minimal submatching starting from σ thus $\sigma \cup \sigma_1$ itself is feasible. First, it is clear that $I_{\sigma_1} - I_{\sigma \cap \sigma'} = I_{\sigma \cup \sigma_1} - I_\sigma$ and $H_{\sigma_1} - H_{\sigma \cap \sigma'} = H_{\sigma \cup \sigma_1} - H_\sigma$. Next, $\sigma \cup \sigma_1$ is an element in $\{\tilde{\sigma} \supset \sigma : H_{\tilde{\sigma}} - H_\sigma \subset c^{-1}_\sigma(I_{\tilde{\sigma}} - I_\sigma)\}$ since

$$\begin{aligned} H_{\sigma \cup \sigma_1} - H_\sigma = H_{\sigma_1} - H_{\sigma \cap \sigma'} &\subset c^{-1}_{\sigma \cap \sigma'}(I_{\sigma_1} - I_{\sigma \cap \sigma'}) \\ &\subset c^{-1}_\sigma(I_{\sigma_1} - I_{\sigma \cap \sigma'}) = c^{-1}_\sigma(I_{\sigma \cup \sigma_1} - I_\sigma), \end{aligned}$$

where the second $\subset$ follows from the consistency. Third, $\sigma \cup \sigma_1$ is also a minimal element because σ_1 is already a minimal submatching starting from $\sigma \cap \sigma'$. Therefore, By Definition 2, $\sigma \cup \sigma_1$ is a feasible submatching starting from σ.

Finally, this procedure can be inductively applied to show that for any $\ell \leq k$, $\sigma \cup \sigma_\ell$ is a minimal matching in $\{\tilde{\sigma} \supset \sigma_{\ell-1} \cup \sigma : H_{\tilde{\sigma}} - H_{\sigma_{\ell-1} \cup \sigma} \subset c^{-1}_{\sigma_{\ell-1} \cup \sigma}(I_{\tilde{\sigma}} - I_{\sigma_{\ell-1} \cup \sigma})\}$ thus $\sigma \cup \sigma_\ell$ is feasible. Therefore, $\sigma \cup \sigma' = \sigma \cup \sigma_k$ is also a feasible submatching.

A.2 Proof of Theorem 1

Fix an inheritance structure $\{c_\sigma\}$ and let φ denote the hierarchical exchange rule. Fix a preference profile P. Assume that $S_1, \cdots, S_T$ are sets of agents matched in Steps $1, \cdots, T$, respectively, in the hierarchical exchange procedure under P.

Sufficiency. From Pápai (2000), φ is Pareto efficient and strategy-proof. We only need to show that φ is also individually rational. For all $i_1 \in S_1$, individual rationality is satisfied simply because i receives her most favorite house under φ. For any agent $i_2 \in S_2$, she receives her most favorite house among the remaining houses after agents in

S_1 are removed with their assignments, and the contingent endowment of i_2 is a subset of all remaining houses after this removal, her individual rationality is also satisfied. The individual rationality of other students can be shown inductively.

Necessity. Let ψ be any mechanism that is individually rational, Pareto efficient and strategy-proof. We need to show that ψ is φ. We first show by contradiction that for all $i \in S_1$, $\varphi_i(P) = \psi_i(P)$. For this part, we use techniques in the proof of Theorem 1 of Abdulkadiroğlu et. al. (2013). Without loss of generality, suppose agent $1 \in S_1$ and $\varphi_1(P) \neq \psi_1(P)$. Then the underlying top trading cycle in the hierarchical exchange rule containing agent 1 must contain another agent. Otherwise, the favorite house of agent 1, $\varphi_1(P)$, is in her own initial endowment $c_{\emptyset}^{-1}(1)$. Since $\psi(P)$ is individual rational, it implies that $\psi_1(P)$ is at least preferred by agent 1 to any house in her initial endowment. As a result, $\psi_1(P) = \varphi_1(P)$. We have a contradiction.

Without loss of generality, let $1 \to h_k \to k \to h_{k-1} \to k-1 \to \cdots \to h_1 \to 1$ be the underlying top trading cycle containing agent 1 in the hierarchical exchange rule, where $k \geq 2$. It is clear that for any agent $i = 1, \cdots, k, h_i \in c_{\emptyset}^{-1}(i)$ and $\varphi_1(P) = h_k, \varphi_2(P) = h_1, \cdots, \varphi_k(P) = h_{k-1}$. Since h_k is 1's most favorite house under P_1 and $\varphi_1(P) = h_k \neq \psi_1(P)$, it must be the case that $h_k P_1 \psi_1(P)$. Let $P_1' : h_k, h_1, \emptyset$. That is, under this preference P_1', 1 ranks h_k at top, h_1 second, and does not accept any house else. Let $Q^1 = (P_1', P_{-1})$. It is obvious that $\varphi(P) = \varphi(Q^1)$; that is, the hierarchical exchange rule outcomes under P and Q^1 coincide. We next show that $\psi_1\left(Q^1\right) = h_1$. Indeed, since $\psi\left(Q^1\right)$ is individual rational and $h_1 \in c_{\emptyset}^{-1}(1)$, we know that $\psi_1\left(Q^1\right) R_1' h_1$. As a result, $\psi_1\left(Q^1\right)$ must be either h_1 or h_k. However, $\psi_1\left(Q^1\right)$ can not be h_k. Otherwise, under ψ, at (P_1, P_{-1}), agent 1 will misreport P_1' to obtain h_k, which is strictly better than $\psi_1(P_1, P_{-1})$. This contradicts with the strategy-proofness of ψ.

Since $\psi_1\left(Q^1\right) = h_1, \psi_2\left(Q^1\right) \neq h_1$. As a result, $\varphi_2\left(Q^1\right) = \varphi_2(P) = h_1 \neq \psi_2(Q^1)$.

Consequently, $\varphi_2(Q^1) P_2 \psi_2\left(Q^1\right)$. Let $P_2' = h_1, h_2, \emptyset$, and let $Q^2 = \left(P_1', P_2', P_{-\{1,2\}}\right)$. Again, $\varphi(P) = \varphi\left(Q^1\right) = \varphi\left(Q^2\right)$, that is, the hierarchical exchange rule outcomes under P, Q^1 and Q^2 coincide. Similar arguments as in the previous paragraph can be used to verify that $\psi_2(Q^2) = h_2$. Furthermore, this procedure can be continued inductively for every agent in this top trading cycle containing 1. To summarize, there is a preference profile $Q^k = \left(P_1', P_2', \cdots, P_k', P_{-\{1,2,\cdots,k\}}\right)$, where $P_i' = (h_{i-1}, h_i, \emptyset)$ for all $i = 2, \cdots, k$, such that $\psi_i\left(Q^k\right) = h_i$ for all $i = 2, \cdots, k$. Note that at $(P_1', P_2', \ldots, P_k', P_{-\{1,2,\ldots,k\}})$, agents in $\{1, 2, \ldots, k\}$ can trade their assignments under ψ in the same cycle as in $\varphi(P)$ to make everybody strictly better off under Q^k. This violates the Pareto efficiency of ψ.

We have thus shown that for all agent $i \in S_1, \varphi_i(P) = \psi_i(P)$. Now consider any agent $i' \in S_2$ and the cycle that i' is traded in φ–the hierarchical exchange rule. Suppose $h_{i'}$ is the house that agent i' uses to trade under φ. Then under φ, after the removal of the submatching σ_{S_1} associated with $S_1, h_{i'}$ is endowed to i' according to $\{c_\sigma\}$. Apparently, σ_{S_1} is feasible. Now consider the matching $\psi(P)$. Since for all $i \in S_1, \varphi_i(P) = \psi_i(P), \sigma_{S_1}$ is also a submatching of $\psi(P)$. That is, σ_{S_1} is a feasible submatching of $\psi(P)$ that excludes i'. As a result, at $\psi(P), h_{i'}$ must belong to $\omega(i' | \psi(P))$–the contingent endowment of i' at $\psi(P)$.

Denote by σ_1 the submatching in Step 1 in the hierarchical exchange procedure (associated with agents in S_1). As shown above, for all agents $i \in S_1, \psi_i(P) = \varphi_i(P)$, so σ_1 is a feasible submatching both in $\psi(P)$ and in $\varphi(P)$. Therefore, in the hierarchical exchange procedure, for any agent j removed in Step 2, her house h_j used to be traded (or taken by herself) must be in her endowment after the submatching σ_1 is removed. Following the same procedure as before, one can show that for all $j \in S_2, \varphi_j(P) = \psi_j(P)$. The rest of the proof follows from induction by proving $\varphi_j(P) = \psi_j(P)$ for all agent $j \in S_3, \cdots, S_T$. We thus complete the proof of $\varphi(P) = \psi(P)$.

A.3 Proof of Theorem 2

For any house allocation problem with a consistent inheritance structure $(P, \{c_\sigma\})$, denote the hierarchical exchange outcome by μ. We first show that μ is a core allocation and then show it is the unique core allocation.

Suppose instead μ is not in the core. Then by the definition of core, the matching μ is blocked by a coalition $B \subset I$ via another matching ν. Let $\nu(B) = \{\nu(i) : i \in B\}$. For each $k = 1, \ldots, T$, let S_k be the set of agents removed at Step k of the hierarchical exchange procedure. It is clear that the S_k's form a partition of I. We divide the rest of the proof into three parts.

Part (i). Denote by $A_0 = \{j \in I : \nu(B) \cap c_\emptyset^{-1}(j) \neq \emptyset\}$ the set of agents who are initially endowed with houses in $\nu(B)$. Since $\{c_\emptyset^{-1}(j) : j \in I\}$ form a partition of H, A_0 is nonempty. By the definition of core, under matching ν, every agent in B must not be matched with any house in the contingent endowment of any agent in B^c. As a result, we must have $A_0 \subset B$. Let k_1 be the earliest step of the hierarchical exchange procedure such that some agent in A_0 is matched and removed; formally, $k_1 = \min\{k : S_k \cap A_0 \neq \emptyset\}$.

Claim 1. *The following holds:*

(a) *For any $i \in S_{k_1} \cap A_0$, $\mu(i) = \nu(i)$;*

(b) *If $i \in S_{k_1} \cap A_0$ and C is the set of agents in the top trading cycle that contains i in the hierarchical exchange procedure, then $C \subset S_{k_1} \cap A_0$;*

(c) *Denote by ν_1 the submatching of ν restricted to $S_{k_1} \cap A_0$. Then ν_1 is a feasible submatching at $c_\emptyset$.*

Proof of Claim 1. (a) For any agent $i \in S_{k_1} \cap A_0 \subset B$, we first show that $\nu(i) = \mu(i)$. Suppose not, it follows from $\nu(i) R_i \mu(i)$ that $\nu(i) P_i \mu(i)$. It is clear that $\nu(i) \notin c_\emptyset^{-1}(i)$ because otherwise the individual rationality of agent i is violated at the hierarchical exchange outcome μ. Let j be the agent such that $\nu(i) \in c_{\emptyset}^{-1}(j)$. By the definition of A_0 and the fact that $i \in A_0 \subset B$, $j \in A_0$. Moreover, since $\nu(i) P_i \mu(i)$, it must be the case that in the hierarchical exchange procedure, $\nu(i)$ has been traded by j in a step, say k_0, which is strictly earlier than Step k_1. As a result, $j \in A_0 \cap S_{k_0}$ and $k_0 < k_1$, which are in contradiction with the definition of k_1.

(b) Suppose $i \in S_{k_1} \cap A_0$ and C is the set of agents in the top trading cycle that contains i. Suppose $\nu(i) = \mu(i) \in c_{\emptyset}^{-1}(j')$ for an agent j', then since $i \in B$, $j' \in A_0$. By the definition of k_1, agent j' can not belong to S_k. Moreover, j' does not belong to S_k for any $k > k_1$ because agent i cannot obtain $\mu(i)$ in the hierarchical exchange procedure in an earlier step in which agent j' is matched. Therefore, agent j' must be in the same trading cycle as agent i in this procedure and hence $j' \in S_{k_1}$. Next, we repeat the same argument to show that $j' \in C$ and $\nu(j') = \mu(j')$, and further prove $C \subset S_{k_1} \cap A_0$.

(c) Let $C_1, \cdots, C_L$ be a partition of $S_{k_1} \cap A_0$ such that for each $\ell = 1, \cdots, L$, $C_\ell \subset S_{k_1}$ is the set of agents in a top trading cycle at Step k_1 in the inheritance exchange procedure. We next show that, for any $\ell = 1, \cdots, L$, the submatching restricted to the subset C_ℓ, denoted it by ν_{C_ℓ}, is a feasible submatching at $c_\emptyset$. In fact, let $C_\ell = \{i_1 \to i_2 \to \cdots \to i_\ell\}$. It follows form part (a) and (b) that, without loss of generality, $\nu(i_1) \in c_{\emptyset}^{-1}(i_2), \nu(i_2) \in c_{\emptyset}^{-1}(i_3) \cdots, \nu(i_\ell) \in c_{\emptyset}^{-1}(i_1)$. It is clear that ν_{C_ℓ} is a minimal element in $\{\sigma : H_\sigma \subset c_{\emptyset}^{-1}(I_\sigma)\}$. By Definition 2, ν_{C_ℓ} is a feasible submatching at $c_\emptyset$. Finally, since $C_1, \cdots, C_k$ are disjoint subsets and each ν_{C_ℓ} is a feasible submatching, it follows from Proposition 1, part (ii), that $\nu_1 = \nu_{C_1} \cup \cdots \cup \nu_{C_k}$ is also a feasible submatching at $c_\emptyset$. □

Part (ii). According to Claim 1, ν_1, the submatching of ν restricted to $S_{k_1} \cap A_0$, is a feasible submatching. Let $B_1 = B - I_{\nu_1} = B - S_{k_1} \cap A_0$. It is clear that B_1 is a strict subset of B and $I_{\nu_1} \subset A_0$. This set $B_1 \neq \emptyset$ because otherwise there is no agent in B who strictly prefers the house matched in ν to the house matched in μ, which contradicts with the hypothesis that ν dominates μ through the coalition B.

Let $A_1 = \{j \in \bar{I}_{\nu_1} : \nu(B_1) \cap c_{\nu_1}^{-1}(j) \neq \emptyset\}$ be the set of agents who endow houses in $\nu(B_1)$ after the feasible submatching ν_1 is removed. Agents who are initially endowed with houses in $\nu(B)$ but not matched at ν_1 belong to A_1, i.e., $A_0 - I_{\nu_1} \subset A_1$. In fact, for any agent $j \in A_0 - I_{\nu_1}$, by the definition of k_1, agent j is not matched at steps $1, \cdots, k_1$ in the hierarchical exchange rule, i.e., $j \notin S_1 \cup \cdots \cup S_{k_1}$. Note that the endowment of agent j after Step k_1 in the inheritance exchange procedure can not contain any house matched to agents in the first k_1 steps. As a result, $\nu(S_{k_1} \cap A_0) \cap c_{\nu_1}^{-1}(j) = \emptyset$. Since $j \in A_0$, $\nu(B) \cap c_{\emptyset}^{-1}(j) \neq \emptyset$, it follows by consistency that $\nu(B) \cap c_{\nu_1}^{-1}(j) \neq \emptyset$. Therefore, it follows that $\nu(B - S_{k_1} \cap A_0) \cap c_{\nu_1}^{-1}(j) \neq \emptyset$, i.e., $j \in A_1$.

Let $k_2 = \min\{k : S_k \cap A_1 \neq \emptyset\}$. It is clear that k_2 is well defined and $k_2 > k_1$. Similar arguments in the proof of Claim 1 can be used to prove the following claim.

Claim 2. *The following holds:*

(a) *For any $i \in S_{k_2} \cap A_1$, $\mu(i) = \nu(i)$;*

(b) *Let C be the set of agents in the top trading cycle containing an agent in $S_{k_2} \cap A_1$ in the hierarchical exchange procedure, then $C \subset S_{k_2} \cap A_1$;*

(c) *Denote by ν_2 the submatching of ν restricted to $S_{k_2} \cap A_1$. Then $\nu_1 \cup \nu_2$ is a feasible submatching.*

Part (iii). The arguments in Parts (i) and (ii) can be continued inductively and this procedure will stop in finite steps since B is a finite set and a nonempty subset of B is removed in every step. However, as stated in Claim 1 (a) and its analogue, any agent in B must be matched with the same house in μ and ν. This contradicts with the fact that ν dominates μ through coalition B. We thus complete the proof that the hierarchical exchange rule outcome μ is in the core of the house allocation problem with inheritance.

Finally, we prove that the top trading cycle outcome μ is the unique core allocation. Suppose instead $\nu \neq \mu$ is another core allocation. Let k be the smallest number such that $\{j : \nu(j) \neq \mu(j)\} \cap S_k \neq \emptyset$. It is routine to check that ν is blocked by the coalition $S_1 \cup \cdots \cup S_k$ via μ. We thus complete the proof of Theorem 2.

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