Interim Partially Correlated Rationalizability

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Abstract

We formalize a solution concept called interim partially correlated rationalizability (IPCR), which was implicitly discussed in both Ely and Peski (2006) and Dekel et al. (2007). IPCR allows for interim correlations, i.e., correlations that depend on opponents' types but not on the state of nature. As a direct extension of Ely and Peski's main result, we show that hierarchies of beliefs over conditional beliefs are necessary and sufficient for the identification of IPCR. We use new proof techniques that better illustrate the connection between higher order beliefs and interim rationalizability.

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1 Introduction

We follow the literature that study the connection between solution concepts and information modeling in games with incomplete information. Conventionally, modelers choose hierarchies of beliefs about payoffs as the primitives of the game and use Harsanyi type spaces (Harsanyi, 1967-1968) to model them. Solution concepts are then defined on type spaces. However, as is commonly known in the literature, under a fixed payoff structure of a game, type spaces that represent the same set of hierarchies of beliefs may give different Bayes Nash equilibrium predictions.[1]

Two approaches are taken to restore the connection between solution concepts and information. Dekel et al. (2007) (hereafter, DFM) define interim correlated rationalizability (hereafter, ICR) and show that the conventional Mertens-Zamir hierarchy of beliefs of a type is sufficient for identifying the set of ICR actions of that type.[2] In their definition, a player may conjecture that her opponents' types, states of nature and opponents' actions could be arbitrarily correlated, as long as the correlation is consistent with her belief in the type space. In a parallel work, Ely and Peski (2006) (hereafter, EP) study interim independent rationalizability (hereafter, IIR) for two-player games, in which each player assumes that the opponent's action correlates with state of nature only through the opponent's type. They introduce Δ-hierarchy of beliefs which contains weakly richer information than conventional hierarchy of beliefs at any type and show that it is necessary and sufficient for IIR.

When there are more than two players, a player could conjecture that there are

[1] For example, any correlated equilibrium in a complete information game can be viewed as a type space. On all such type profiles, players have common knowledge of the game. However, there may be action profiles played in Bayes Nash equilibrium but are not Nash equilibrium of the complete information game.

[2] The necessity condition is given in Dekel et al. (2006).

correlations among other players' strategies.[3] Unlike in games with complete information where the correlation takes a natural form, for games with incomplete information, there are multiple possibilities for correlations, depending on which type of information is given as the primitive. In particular, there are multiple ways of extending EP's formulation to games with more than two players, as is also discussed in Ely and Peski (2006), section 7.2.

Following EP, we define interim partially correlated rationalizability (hereafter, IPCR) for n-player games by considering conjectures that involve only interim stage correlations. We assume that players view opponents' actions as type-contingent variables, and that a player's conjecture over opponents' actions and state of nature is induced by her belief in the type space together with a type-correlated strategy of the opponents'. A type-correlated strategy of the opponents' maps each profile of their types to a probability measure on their action profiles. If we take the agent-normal-form view of a type space, i.e., if we view each type of a player as an agent of that player, then the correlation in IPCR is similar to that permitted in the definition of correlated rationalizability for complete information games played by agents. We may also view the correlation we permit as interim correlation, while view that permitted by DFM as ex post correlation.

Our main result is a direct extension of EP's main theorem, in which we show that two types have the same interim partially correlated rationalizable behavior if and only if they have the same Δ-hierarchy of beliefs. In the n-player environment, Δ-hierarchy of beliefs is defined as the hierarchy of beliefs over conditional beliefs, where each conditional belief of a player is her belief over states of nature given a profile of her opponents' types. This result justifies the definition of IPCR, and also provides another connection

[3]Such correlations are familiar to us and are well-studied for games with complete information under correlated equilibrium (Aumann, 1974, and Aumann, 1987) and correlated rationalizability (Bernheim, 1984, Pearce, 1984, and Brandenburger and Dekel, 1987).

between solution concepts and information.

Why do we need richer information to identify IPCR? Similar to the case of IIR, it is because conventional hierarchy of beliefs fails to capture certain correlations that potentially affect IPCR. We illustrate this with a simple but non-degenerate three-player game with incomplete information. This example also suggests the necessity of beliefs over conditional beliefs.[4]

Example 1. Let $\Theta = \{+1, -1\}$ be the set of states of nature. Consider two type spaces that both model common knowledge of equal probability on $\theta = +1$ and $\theta = -1$. In type space T, $T_1 = T_2 = T_3 = \{*\}$, and the common prior is $\mu[\theta = +1] = \mu[\theta = -1] = \frac{1}{2}$. In type space $\hat{T}$, $\hat{T}_1 = \{*\}$, $\hat{T}_2 = \hat{T}_3 = \{+1, -1\}$, and the common prior $\hat{\mu}$ on $\hat{T}_2 \times \hat{T}_3 \times \Theta$ is given by

$T_2 \backslash T_3$	$+1$	-1
$+1$	$\frac{1}{4}$	0
-1	0	$\frac{1}{4}$

$\theta = +1$

$T_2 \backslash T_3$	$+1$	-1
$+1$	0	$\frac{1}{4}$
-1	$\frac{1}{4}$	0

$\theta = -1$

Suppose player 2 and player 3 are trying to coordinate their actions on the right state of nature and their payoffs (given in the tables below) do not depend on player 1's action.

	a_3	b_3
a_2	$1, 1$	$0, 0$
b_2	$0, 0$	$1, 1$

$\theta = +1$

	a_3	b_3
a_2	$0, 0$	$1, 1$
b_2	$1, 1$	$0, 0$

$\theta = -1$

Player 1, instead, chooses between *Bet* and *Not Bet*. By choosing *Bet*, she receives a

[4] This example incorporates elements from the motivating examples of both Ely and Peski (2006) and Liu (2015).

payoff of 1 when the other two successfully coordinate on the right state θ; otherwise she receives 0. By choosing *Not Bet*, she always receives $\alpha \in (\frac{1}{2}, 1)$.

We use IPCR as the solution concept. When the type space is T, for example, in Player 1's conjecture, players 2 and 3 can play any type-correlated strategy $\sigma_{2,3} : T_2 \times T_3 \to \Delta(\{a_2, b_2\} \times \{a_3, b_3\})$,[5] but their strategies cannot directly depend on θ, the true state of nature. Note that at type space T, for any conjecture of player 1, her expected payoff from choosing *Bet* is $\frac{1}{2}$, less than α, her payoff from choosing *Not Bet*. Thus, the action *Bet* is not rationalizable for her at type space T. When the type space is $\hat{T}$, however, player 1 may justifiably conjecture that both opponents choose a at type $+1$ and choose b at type -1, thus successfully coordinate at both states of nature. Under this conjecture, player 1 receives $1 > \alpha$ by choosing *Bet*. That is, at type space $\hat{T}$, *Bet* becomes rationalizable for player 1.

Hence we see that at both type spaces, player 1 has the same conventional hierarchy of beliefs, but has different sets of IPCR actions. Essentially, player 2 and 3 can coordinate better at type space $\hat{T}$ because their types in $\hat{T}$ involve more correlation with θ.[6] We show explicitly in Example 2 that such correlation is captured by player 1's hierarchy of beliefs over conditional beliefs.

In the proof of our main theorem, we use techniques that are more transparent and make the connection between hierarchy of beliefs and interim rationalizability easier to understand. More specifically, in proving the sufficiency of Δ-hierarchy of beliefs for IPCR, we adapt DFM's proof of proving the sufficiency of conventional hierarchy of beliefs for ICR. From the proof, we see inductively that the k-th order belief of a player de-

[5]For any complete separable metric space X, ΔX is the space of Borel probability measures on X.

[6]In fact, type space $\hat{T}$ can be generated through type space T using a simple state-dependent correlating device, which does not change players' hierarchies of beliefs (Liu, 2015). However, if such a device is not a partially correlating device, then it changes players' hierarchies of beliefs over conditional beliefs (Tang, 2015). Please see Section 4 for more discussion.

termines her k-th level feasible conjectures, hence determines her k-th level rationalizable actions. In proving the necessity part, we use a new technique that involves inductively constructing betting games to separate beliefs in the hierarchy. In a betting game that is constructed to separate a player's k-th order beliefs, we let her bet on opponents' actions which are determined by their $(k-1)$-th order beliefs. Although we use intermediate results from EP, the technique itself is genuinely different from those in existing proofs.

The rest of the paper is organized as follows. We present the model and basic definitions in Section 2, and the main result in Section 3. In Section 4, we discuss the relationship of this paper with Liu (2015), the payoff equivalence between IPCR and the Bayesian solution–a notion of correlated equilibrium proposed by Forges (1993), and other related literature.

2 Model

We begin with some notations. For any Polish (or equivalently, complete separable metric) space X, let ΔX denote the space of Borel probability measures on the Borel σ-algebra of X endowed with the weak*-topology. Let the product of two Polish spaces be endowed with the product Borel σ-algebra. Let $\operatorname{supp} \mu$ be the support of a Borel probability measure μ, i.e., the smallest closed set with probability 1 under μ. For any measure $\mu \in \Delta(X \times Y)$, denote by $\operatorname{marg}_X \mu$ the marginal distribution of μ on X. For any measure $\mu \in \Delta X$ and integrable function $f : X \to R$, denote by $\mu[f]$ the expectation of f under μ.

We study n-player games with incomplete information. The set of players is $N = \{1, 2, ..., n\}$. For each $i \in N$, let $-i$ denote the set of i's opponents. Players play a game in which the payoffs are uncertain and parameterized by a finite set Θ. Each element $\theta \in \Theta$ is called a state of nature. For each $i \in N$, denote by A_i the finite set of actions for player i, and $A \equiv \times_{i \in N} A_i$ the set of action profiles. A (strategic form) game is a profile $G = (g_i, A_i)_{i \in N}$, where for each $i \in N$, $g_i : A \times \Theta \to [-M, M]$ is a bounded payoff function for player i. Let the set of finite bounded games be denoted by $\mathcal{G}$.

A type space over Θ is a tuple $T = (T_i, \pi_i)_{i \in N}$, where for each i, T_i is a compact metric space of types for player i and $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$ is a measurable mapping that describes player i's belief over opponents' types and states of nature for any type of player i. A strategy of player i is a mapping $\sigma_i : T_i \to \Delta A_i$. Let $\sigma = (\sigma_i)_{i \in N}$ be a strategy profile, and with a little abuse of notation, let $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ be a type-correlated strategy of the opponents.

Throughout, given arbitrary $x \in X$ and $y \in Y$, we use the notation $\pi_i(x)[y]$ to denote player i's belief about y conditional on x. More precisely, the object in the round bracket always denotes the object that player i conditions on, and the object in the square bracket always denotes the object that player i assigns probability to.

2.1 Interim partially correlated rationalizability

Player i's conjecture at t_i is a joint distribution $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ on opponents' types, states of nature and opponents' actions such that $\text{marg}_{T_{-i} \times \Theta} v = \pi_i(t_i)$. Let $m^v[(\theta, a_{-i})|t_i] \equiv \int_{T_{-i}} v[(dt_{-i}, \theta, a_{-i})]$ denote the marginal probability of v at (θ, a_{-i}). In short, let $m^v = \text{marg}_{\Theta \times A_{-i}} v$.

Rationalizability can be defined in many equivalent approaches; we adopt the iterative elimination of never best response actions procedure. At each round, an action a_i of player i survives the elimination at type t_i if it is a best response to some conjecture $v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ that can be induced from $\pi_i(t_i)$ and a type-correlated strategy σ_{-i} of the opponents, and at the type-correlated strategy, the opponents only play action profiles in A_{-i} that survived previous rounds of elimination.

Definition 1. *Fix a game G and a type space T. For any $t_i \in T_i$, let $R^T_{i,0}(t_i|G) \equiv A_i$. An action is level-k rationalizable at t_i if $a_i \in R^T_{i,k}(t_i|G)$, where*

$$R^T_{i,k}(t_i|G) = \left\{ a_i \in A_i : \begin{array}{l} \text{there exist } v \in \Delta(T_{-i} \times \Theta \times A_{-i}) \text{ and a measurable type-correlated} \\ \text{strategy } \sigma_{-i} : T_{-i} \to \Delta A_{-i} \text{ such that:} \\ (1)\ (t_{-i}, \theta, a_{-i}) \in \operatorname{supp} v \Rightarrow a_{-i} \in (R^T_{j,(k-1)}(t_j)|G)_{j \neq i} \\ (2)\ a_i \in \arg\max_{a'_i \in A_i} \sum_{\theta, a_{-i}} g_i((a'_i, a_{-i}), \theta) m^v[(\theta, a_{-i})|t_i] \\ (3)\ m^v[(\theta, a_{-i})|t_i] = \int_{T_{-i}} \sigma_{-i}(t_{-i})[a_{-i}] \cdot \pi_i(t_i)[(dt_{-i}, \theta)] \end{array} \right\},$$

and $R^T_i(t_i|G) = \bigcap_{k=1}^{\infty} R^T_{i,k}(t_i|G)$. Actions in $R^T_i(t_i|G)$ are said to be interim partially correlated rationalizable (IPCR) at type t_i.

Due to the same argument as that of Ely and Peski (2006) on the non-emptiness of IIR, $R^T_i(t_i|G)$ is non-empty. Hereafter, we suppress the notation G in $R^T_i(t_i|G)$ unless it is necessary for clarity.

The type-correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ deserves some clarification. We are not assuming that the opponents are sharing information with each other and are thus playing in a coordinated fashion. Instead, we take the view that the correlation may come from possibly correlated type-contingent extraneous signals that the opponents receive, or from player i's ignorance over the opponents' beliefs about each other's action (Aumann, 1987, section 6).

2.2 Hierarchies of beliefs

We first present Mertens and Zamir (1985)'s standard formulation of hierarchies of beliefs (see also Brandenburger and Dekel, 1993), and based on that present Ely and Peski (2006)'s construction of Δ-hierarchies of beliefs. For convenience, we call Mertens-Zamir hierarchy of beliefs the conventional hierarchy of beliefs.

Let $X_0 = \Theta$, and for $k \geq 1, X_k = X_{k-1} \times (\Delta(X_{k-1}))^{n-1}$. Let $h^1(t_i) = \text{marg}_{\Theta}\, \pi_i(t_i)$, which is player i's belief over Θ at type t_i. For each $k \geq 1$, let $h^k(t_i)[S] = \pi_i(t_i)[\{(\theta, t_{-i}) : (\theta, (h^l(t_j), j \neq i)_{1 \leq l \leq k-1}) \in S\}]$, for any measurable subset $S \subseteq X_{k-1}$. In the construction, $h^k(t_i) \in \Delta(X_{k-1})$ represents player i's k-th order belief at t_i. Player i's conventional hierarchy of beliefs at type t_i is defined as the profile $h(t_i) = (h^1(t_i), ..., h^k(t_i), ...)$.

A Δ-hierarchy of beliefs describes a player's belief and higher-order beliefs over the set of conditional beliefs on states of nature. The concept was introduced by Ely and Peski (2006) in their study of interim independent rationalizability. We begin with defining conditional beliefs. Given belief $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$, the conditional belief of type t_i over Θ, conditioning on the opponents' types being t_{-i}, is $\pi_i(t_i)(t_{-i}) \in \Delta\Theta$, also written as $\pi_i(t_i, t_{-i})$.[7] For any type space T, let $B_i(t_i) = \{\pi_i(t_i, t_{-i}) \in \Delta\Theta : t_{-i} \in T_{-i}\}$ be the set of possible conditional beliefs at t_i. Type t_i's belief over T_{-i} then induces a belief over $\Delta\Theta$: for any measurable subset $S \subseteq \Delta\Theta, \pi_i(t_i)[S] = \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) \in S\}]$.

Now we can define Δ-hierarchy of beliefs at t_i by treating the set of conditional beliefs $\Delta\Theta$ as the set of basic uncertainties. Let the first-order belief of a player be her belief over the set of conditional beliefs, the second-order belief be her belief over opponents' beliefs over the set of conditional beliefs, and so on.

Formally, for any type space $T = (T_i, \pi_i)_{i \in N}$ on Θ, we can transform it into a type

[7] The conditional belief exists whenever Θ is a Polish space.

space $T^{\Delta} = (T_i, \pi_i^{\Delta})_{i \in N}$ on $\Delta\Theta$. In the new type space, players' types are unchanged, and $\pi_i^{\Delta}(t_i) \in \Delta(T_{-i} \times \Delta\Theta)$ is given by

$$\pi_i^{\Delta}(t_i)[S] = \pi_i(t_i)[\{t_{-i} : (t_{-i}, \pi_i(t_i, t_{-i})) \in S\}],$$

for any measurable subset $S \subseteq \Delta(T_{-i} \times \Delta\Theta)$. Let the conventional hierarchy of beliefs at t_i in the type space T^{Δ} be denoted by $h(t_i|T^{\Delta})$.

Definition 2. *For any type space T, for any $k \geq 1$, let the k-th order Δ-hierarchy of beliefs at $t_i \in T_i$ be $h^k(t_i|T^{\Delta})$ and denote it by $\delta^k(t_i)$. Also, let $\delta(t_i) = (\delta^1(t_i), ..., \delta^k(t_i), ...)$ denote the Δ-hierarchy of beliefs at t_i.*

By definition, $\delta(t_i) = h(t_i|T^{\Delta})$. For player i, we use δ_{-i} to denote the profile of opponents' Δ-hierarchies of beliefs.

Example 2. We revisit Example 1 to show that at type $t_1 = *$, player 1 has different Δ-hierarchies of beliefs in type spaces T and $\hat{T}$.

In the type space T, conditional on $(t_2, t_3) = (*, *)$, player 1's conditional belief on Θ is equal probability on each state, and her first-order belief (in the Δ-hierarchy of beliefs) is certainty about the equal probability.

In the type space $\hat{T}$, conditional on $(\hat{t}_2, \hat{t}_3) = (+1, +1)$ or $(-1, -1)$, player 1's conditional belief on Θ is certainty about $\theta = +1$. Similarly, conditional on $(\hat{t}_2, \hat{t}_3) = (+1, -1)$ or $(-1, +1)$, her conditional belief is certainty about $\theta = -1$. Given her belief on $\hat{T}_2 \times \hat{T}_3$, her first-order belief (in the Δ-hierarchy of beliefs) is equal probability on each certainty.

3 Rationalizability and hierarchies of beliefs

The following result shows that two types provide the same IPCR prediction if and only if they have the same Δ-hierarchy of beliefs.

Theorem 1. *If $t_i \in T_i, t_i' \in T_i'$, then $\delta(t_i) = \delta(t_i')$ if and only if $R_i^T(t_i|G) = R_i^{T'}(t_i'|G), \forall G \in \mathcal{G}$.*

The theorem is a direct extension of Ely and Peski (2006)'s main result (section 4, Theorem 2) from two-player games to n-player games. The sufficiency of Δ-hierarchy of beliefs says that it has incorporated all the information modeled in a type that would influence the type's IPCR behavior. The necessity condition, on the other hand, states that all the information characterized by Δ-hierarchy of beliefs matter for IPCR. If two types differ in any order of beliefs, there must be a game in which the types induce different sets of rationalizable actions.

Unlike Ely and Peski (2006), who use abstract structures in the proof of the theorem, we use more straightforward techniques, especially for the part of necessity. We believe that these techniques will better illustrate the connection between hierarchies of beliefs and interim rationalizability.

In the proof of sufficiency, we closely follow Dekel et al. (2007), in which they show that conventional hierarchy of beliefs is sufficient for identifying interim correlated rationalizability. The intuition is as follows. Suppose there two types $t_i \in T_i$ and $t_i' \in T_i'$. For each conjecture v that is feasible at t_i, it is supposed to be supported by some type-correlated strategy $\sigma_{-i} : T_{-i} \to \Delta A_{-i}$ of the opponents. We show that as long as $\delta(t_i) = \delta(t_i')$, we can always construct $\sigma'_{-i} : T'_{-i} \to \Delta A_{-i}$ such that the conjecture v' generated by σ'_{-i} has the same marginal distribution on $\Theta \times A_{-i}$ as v does. More specifically, in the construction of σ'_{-i}, we let all t'_{-i}'s with the same conditional belief $\pi_i'(t_i', t_i') = \beta \in \Delta\Theta$ play the average strategy of the t_{-i}'s with $\pi_i(t_i, t_{-i}) = \beta$.

In the proof of necessity, we use intermediate results from Ely and Peski (2006) and a technique that is inspired by Gossner and Mertens (2001).[8] To be precise, we show that for each k, if $\delta^k(t_i) \neq \delta^k(t'_i)$, then we can construct a betting game $G(\delta^k, \delta'^k)$ in which the two types have different sets of rationalizable actions. The construction is inductive. Suppose we know how to construct games for the case of $(k-1)$ and $\delta^k(t_i) \neq \delta^k(t'_i)$. In the k-th level games, we let player i play against the opponents who are playing $(k-1)$-th level games. It is a feasible conjecture for player i to believe that the opponents are choosing rationalizable type-correlated strategies that maximize her payoff. However, since types t_i and t'_i have different beliefs on opponents' $(k-1)$-th order beliefs, they have different sets of rationalizable beliefs. This gives us the leeway to construct level k betting games.

The same technique can be slightly adapted to prove the necessity of conventional hierarchy of beliefs for interim correlated rationalizability (Dekel et al., 2006, Lemma 4). We only need to note that suppose we add nature as an independent player into the game, then IPCR is equivalent to interim correlated rationalizability and the information that Δ-hierarchy of beliefs incorporates is reduced to that of conventional hierarchy of beliefs. Of course, we can also prove that result directly with this technique.

[8] Gossner and Mertens (2001) study the value of information in zero-sum games. The idea of proving their Lemma 4 and Lemma 5 involves constructing betting games in which one player bets on the other's action, which depends on the other's first-order belief.

3.1 Proof of sufficiency

Fix a game $G \in \mathcal{G}$. We need to show that if $\delta(t_i) = \delta(t_i')$, then $R_i^T(t_i) = R_i^{T'}(t_i')$. Denote the set of all possible conjectures of player i in the k-th round of the elimination procedure by

$$V_i^k(t_i) = \begin{cases} v \in \Delta(T_{-i} \times \Theta \times A_{-i}) \text{ such that:} \\ (1)(t_{-i}, \theta, a_{-i}) \in \operatorname{supp} v \Rightarrow a_{-i} \in R^T_{-i,(k-1)}(t_{-i}); \\ (2) \int_{T_{-i}} v[(t_{-i}, \theta, a_{-i})] dt_{-i} = \int_{T_{-i}} \pi_i(t_i, t_{-i})[\theta] \sigma_{-i}(t_{-i})[a_{-i}] \pi_i(t_i)[dt_{-i}]. \end{cases}$$

Denote the set of marginals of $V_i^k(t_i)$ on $\Theta \times A_{-i}$ by $\operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i)$. From the definition of rationalizability, the set of marginals on $\Theta \times A_{-i}$ determines the set of justifiable expected payoffs, thus determines the set of rationalizable actions. That is, if $\operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i) = \operatorname{marg}_{\Theta \times A_{-i}} V_i^k(t_i')$, then $R^T_{i,k}(t_i) = R^{T'}_{i,k}(t_i')$.

Step 1. We start with the case of $k = 1$ and then prove the rest inductively. Consider the probability space $(T_{-i}, \pi_i(t_i)[\cdot], \mathcal{T}_{-i})$, where $\pi_i(t_i)[\cdot] \in \Delta T_{-i}$ is the marginal of $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$ over T_{-i}, and $\mathcal{T}_{-i}$ is the usual Borel σ-algebra. View $\pi_i(t_i, \cdot) : T_{-i} \to B_i(t_i) \subseteq \Delta\Theta$ as a random variable on T_{-i}, and denote the σ-algebra generated by it by $\sigma(\pi_i(t_i, \cdot))$. Since T_{-i} is a compact metric space, there exists a regular conditional probability that maps from $T_{-i} \times \mathcal{T}_{-i}$ to $[0, 1]$ given $\sigma(\pi_i(t_i, \cdot))$ (see, for example, Durrett, 2004). Since the conditional probability is $\sigma(\pi_i(t_i, \cdot))$ measurable, by a little abuse of notation, we can write it as $\pi_i(t_i, \cdot) : B_i(t_i) \to \Delta T_{-i}$. Now, the marginal distribution for a given conjecture

$v \in \Delta(T_{-i} \times \Theta \times A_{-i})$ over $\Theta \times A_{-i}$ can be expressed as

$$
\begin{aligned}
m^v &= \int_{T_{-i}} \pi_i(t_i, t_{-i})[\theta]\sigma_{-i}(t_{-i})[a_{-i}]d\pi_i(t_i)[t_{-i}] \\
&= \int_{B_i(t_i)} \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta\}} \pi_i(t_i, t_{-i})[\theta]\sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i, \beta)[dt_{-i}]\delta^1(t_i)[d\beta] \\
&= \int_{B_i(t_i)} \beta[\theta]\pi_i(t_i, \beta)[\sigma_{-i}(t_{-i})[a_{-i}]]\delta^1(t_i)[d\beta]
\end{aligned}
$$

We are ready to construct a conjecture v' for type t'_i such that $m^{v'} = m^v$. Suppose t'_i believes that the opponents are playing the following type-correlated strategy: for any type t'_{-i} such that $\pi'_i(t'_i, t'_{-i}) = \beta$,

$$
\begin{aligned}
\sigma'_{-i}(t'_{-i})[a_{-i}] &= \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta\}} \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i, \beta)[dt_{-i}] \\
&= \pi_i(t_i, \beta)[\sigma_{-i}(t_{-i})[a_{-i}]], \forall a_{-i} \in A_{-i}.
\end{aligned}
$$

Intuitively, t'_i believes that at all types t'_{-i} such that $\pi'_i(t'_i, t'_{-i}) = \beta$, action a_{-i} is played with the average of the probabilities it is played with at types t_{-i} satisfying $\pi_i(t_i, t_{-i}) = \beta$. The marginal distribution of v' on $\Theta \times A_{-i}$ is

$$
\begin{aligned}
m^{v'} &= \int_{T'_{-i}} \pi'_i(t'_i, t'_{-i})[\theta]\sigma'_{-i}(t'_{-i})[a_{-i}]\pi'_i(t'_i)[dt'_{-i}] \\
&= \int_{B_i(t'_i)} \int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi'_i(t'_i, t'_{-i})[\theta]\sigma'_{-i}(t'_{-i})[a_{-i}]\pi'_i(t'_i, \beta)[dt'_{-i}]\delta^1(t'_i)[d\beta] \\
&= \int_{B_i(t'_i)} \beta[\theta] \int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi_i(t_i, \beta)[\sigma_{-i}(t_{-i})[a_{-i}]]\pi'_i(t'_i, \beta)[dt'_{-i}]\delta^1(t'_i)[d\beta] \\
&= \int_{B_i(t_i)} \beta[\theta]\pi_i(t_i, \beta)[\sigma_{-i}(t_{-i})[a_{-i}]]\delta^1(t_i)[d\beta] \\
&= m^v,
\end{aligned}
$$

where the first and second equality are straightforward, the third equality comes from the construction of $\sigma'_{-i}(t'_{-i})[a_{-i}]$, and the fourth equality is due to $B_i(t_i) = B_i(t'_i)$, $\delta^1(t_i) = \delta^1(t'_i)$ and $\int_{\{t'_{-i}:\pi'_i(t'_i,t'_{-i})=\beta\}} \pi'_i(t'_i,\beta)[dt'_{-i}] = 1$.

We have shown that any marginal in $\text{marg}_{\Theta\times A_{-i}} V^1_i(t_i)$ also belongs to $\text{marg}_{\Theta\times A_{-i}} V^1_i(t'_i)$. That is, $\text{marg}_{\Theta\times A_{-i}} V^1_i(t_i) \subseteq \text{marg}_{\Theta\times A_{-i}} V^1_i(t'_i)$. By symmetry, $\text{marg}_{\Theta\times A_{-i}} V^1_i(t'_i) \subseteq \text{marg}_{\Theta\times A_{-i}} V^1_i(t_i)$, and hence $\text{marg}_{\Theta\times A_{-i}} V^1_i(t_i) = \text{marg}_{\Theta\times A_{-i}} V^1_i(t'_i)$. By definition, $R^T_{i,1}(t_i) = R^{T'}_{i,1}(t'_i)$, for all $G \in \mathcal{G}$.

Step 2. We prove inductively for cases of $k > 1$. Suppose $R^T_{i,(k-1)}(t_i) = R^{T'}_{i,(k-1)}(t'_i)$ for all $G \in \mathcal{G}$, and $\delta^k(t_i) = \delta^k(t'_i)$. Denote the support of $\delta^k(t_i)$ and $\delta^k(t'_i)$ as $D^{k-1}(t_i)$ and $D^{k-1}(t'_i)$, respectively. We know instantly that $D^{k-1}(t_i) = D^{k-1}(t'_i)$. Denote a generic element in $D^{k-1}(t_i)$ as $(\beta,\delta^{k-1}_1) \equiv (\beta,(\delta^l)_{1\leq l\leq k-1})$. Similar to step 1, we can express the marginal of any conjecture $v \in \Delta(T_{-i}\times\Theta\times R^T_{-i,(k-1)})$ as

$$\begin{aligned}\underset{\Theta\times R^T_{-i,(k-1)}}{\text{marg}} v &= \int_{D^{k-1}(t_i)} \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta^{k-1}_1(t_{-i})=\delta^{k-1}_1\}} \pi_i(t_i,t_{-i})[\theta] \\ &\qquad \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i,(\beta,\delta^{k-1}_1))[dt_{-i}]\delta^k(t_i)[d(\beta,\delta^{k-1}_1)] \\ &= \int_{D^{k-1}(t_i)} \beta[\theta] \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta^{k-1}_1(t_{-i})=\delta^{k-1}_1\}} \sigma_{-i}(t_{-i})[a_{-i}] \\ &\qquad \pi_i(t_i,(\beta,\delta^{k-1}_1))[dt_{-i}]\delta^k(t_i)[d(\beta,\delta^{k-1}_1)],\end{aligned}$$

where $\pi_i(t_i,(\beta,\delta^{k-1}_1))$ is the conditional belief of t_i over t_{-i} at (β,δ^{k-1}_1). To construct the corresponding $v' \in \Delta(T'_{-i}\times\Theta\times A_{-i})$ for v, for any t'_{-i} such that $\pi'_i(t'_i,t'_{-i}) = \beta, \delta^{k-1}_1(t'_{-i}) = \delta^{k-1}_1(t_{-i})$, let

$$\sigma'_{-i}(t'_{-i})[a_{-i}] = \int_{\{t_{-i}:\pi_i(t_i,t_{-i})=\beta,\delta^{k-1}_1(t_{-i})=\delta^{k-1}_1\}} \sigma_{-i}(t_{-i})[a_{-i}]\pi_i(t_i,(\beta,\delta^{k-1}_1))[dt_{-i}],$$

for all $a_{-i} \in R^T_{-i,(k-1)}$, and 0 otherwise. Again we can check that the induced marginal on $\Theta \times A_{-i}$ of the conjecture v' coincides with that of v. Following the same argument as in step 1, $R^T_{i,k}(t_i) = R^T_{i,k}(t'_i)$, for all $G \in \mathcal{G}$. ■

3.2 Proof of necessity

Assume $\delta(t_i) \neq \delta(t'_i)$. Due to the consistency of Δ-hierarchy of beliefs, we can decompose the proof by discussing cases of $\delta^k(t_i) \neq \delta^k(t'_i), \delta^l(t_i) = \delta^l(t'_i), \forall 1 \leq l \leq k$. That is, in the k-th case, the Δ-hierarchies of beliefs at t_i and t'_i differ starting from the k-th level belief. For each case, we construct a game that separates the types in their IPCR behavior. The construction of games is inductive.

Step 1 ($k = 1$). In the first step we consider the case of $\delta^1(t_i) \neq \delta^1(t'_i)$, i.e., when two types have different beliefs over conditional beliefs. We first present a special case of lemma 5 in Ely and Peski (2006).

Lemma 1. *If $\delta^1(t_i), \delta^1(t'_i) \in \Delta(\Delta\Theta)$ and $\delta^1(t_i) \neq \delta^1(t'_i)$, then there exist a natural number m and a continuous bounded function $\psi : \{1, ..., m\}^{N-1} \times \Theta \to [0, \infty)$ such that for $f : \Delta\Theta \to R$ defined by $f(\beta) = \max_{\mathbf{k} \in \{1,...,m\}^{N-1}} \beta[\psi(\mathbf{k}, \theta)]$, we have*

$$\delta^1(t_i)[f] \neq \delta^1(t'_i)[f].$$

Without loss of generality, suppose $\delta^1(t'_i)[f] < \delta^1(t_i)[f]$. By the linearity of expectation, there exists $\lambda > 0$ such that $\delta^1(t'_i)[\lambda f - 1] < 0 < \delta^1(t_i)[\lambda f - 1]$.

With Lemma 1 we construct a finite game $G_i(\delta^1(t_i), \delta^1(t'_i)) = (u_i, A_i)_{i \in N}$ for player i to separate the behavior at types with first-order belief $\delta^1(t_i)$ and types with first-order belief $\delta^1(t'_i)$. Let $A_i = \{0, 1\}$, and $A_j = \{1, ..., m\}, \forall j \neq i$. Let the payoffs to the opponents be a constant, e.g., $u_j(a_j, a_{-j}, \theta) = 0$, for all a_j, a_{-j}, θ. Let the payoff to player i be

$$u_i(a_i, a_{-i}, \theta) = a_i[\lambda\psi(a_{-i}, \theta) - 1].$$

For two-player games with two payoff states, the payoff matrices can be expressed as follows:

	1	2	$\cdots$	m
0	$0, 0$	$0, 0$	$\cdots$	$0, 0$
1	$\lambda\psi(1, \theta_1) - 1, 0$	$\lambda\psi(2, \theta_1) - 1, 0$	$\cdots$	$\lambda\psi(m, \theta_1) - 1, 0$

θ_1

	1	2	$\cdots$	m
0	$0, 0$	$0, 0$	$\cdots$	$0, 0$
1	$\lambda\psi(1, \theta_2) - 1, 0$	$\lambda\psi(2, \theta_2) - 1, 0$	$\cdots$	$\lambda\psi(m, \theta_2) - 1, 0$

θ_2

With these payoffs, for any other player, all actions in $\{1, ...m\}$ are rationalizable. For player i, playing $a_i = 0$ gives her 0, while the payoff from playing $a_i = 1$ depends on the actions of the opponents and state of nature. It is a feasible conjecture of player i that the opponents are playing accordingly to maximize player i's payoff. Player i's payoff from playing $a_i = 1$ is maximized if the opponents play the following type-correlated strategy:

$$\sigma_{-i}(t_{-i}) \in \arg\max_{\mathbf{k}} \beta[\psi(\mathbf{k}, \theta)], \forall t_{-i} \text{ such that } \pi_i(t_i, t_{-i}) = \beta, \forall \beta \in \Delta\Theta.$$

The maximal payoff is $\delta^1(t_i)[\lambda \max_{\mathbf{k}} \beta[\psi(\mathbf{k}, \theta)] - 1] = \delta^1(t_i)[\lambda f - 1]$. Since player i's payoff from playing 1, $\delta^1(t_i)[\lambda f - 1]$, is greater than the payoff from playing $a_i = 0$, which is 0, $a_i = 1$ is rationalizable at t_i. However, at type t_i', the maximal payoff from playing

$a_i = 1$ is $\delta^1(t_i')[\lambda f - 1] < 0$. Therefore, playing $a_i = 1$ is strictly dominated by playing $a_i = 0$; $a_i = 1$ is not rationalizable at t_i'. That is, $R_i(t_i) \neq R_i(t_i')$.

Step 2 (Induction). To carry out induction, we need a generalized version of Lemma 1, which is also adapted from Ely and Peski (2006). Let $F = \{f : \Delta\Theta \to [0, \infty)$ such that $f(\beta) = \max_{\mathbf{k} \in \{1,...,m\}^{N-1}} \beta[\psi(\mathbf{k}, \theta)]$ for some natural number m and continuous bounded function $\psi : \{1, ..., m\}^{N-1} \times \Theta \to [0, \infty)\}$.

Lemma 2. *The collection of sets $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta)$, one for each $f \in F$, generate the weak*-topology on $\Delta(\Delta\Theta)$. This topology is normal, and therefore any pair of disjoint closed subsets $S, S' \in \Delta(\Delta\Theta)$ can be separated by open sets, and there is a function $f \in F$ such that $\forall \mu \in S$ and $\mu' \in S'$,*

$$\mu[f] \neq \mu'[f].$$

Since the proof to Lemma 2 is a special case of lemma 5 in Ely and Peski (2006), we only sketch the idea here. Let H denote the Hilbert cube $[0, 1]^{\mathbb{N}}$, since $\Delta\Theta$ is a second countable Hausdorff space, there is a mapping $H : \Delta\Theta \to \mathbf{H}$ that embeds $\Delta\Theta$ into $\mathbf{H}$ (Urysohn metrization theorem, cf. Aliprantis and Border (2006), Theorem 3.40). Since H is an embedding, the problem of showing $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta)$ for $f \in F$ generates the weak*-topology on $\Delta(\Delta\Theta)$ transforms into showing that there is a family of continuous functions $f : \mathbf{H} \to R$ such that the collection of sets $\{\mu : \mu[f(h)] < 0\}$ generates the weak*-topology on $\Delta(\mathbf{H})$. Let $F_n' = \{f : [0, 1]^n \to R$ such that $f(h_1, ..., h_n) = \max_{\eta \in \{\eta_1,...,\eta_m\}} \eta \cdot h\}$ for some natural number m and a profile of vectors $\eta_1, ..., \eta_m \in [0, 1]^n$. We can prove that the set $L_n' = \{f - g : f, g \in F_n'\}$ is uniformly dense in the set $C([0, 1]^n)$, and hence the family of functions $\cup_n L_n'$ generates the topology on $\Delta(\mathbf{H})$. Now define $F = \{f : f(\beta) = f'(H(\beta))$ for some $f' \in \cup_n L_n'\}$, we see that $\cup_n L_n'$ corresponds to the image of F from the embedding H. Since the topology is Hausdorff on a compact space, it is normal, therefore any pair of disjoint closed subsets can be separated by two open sets.

By applying Lemma 2, for any pair of disjoint closed subsets of first-order beliefs, we can construct a game that separates them in rationalizability. For any pair of disjoint closed subsets $S, S' \in \Delta(\Delta\Theta)$, there is a game $G(S, S')$ such that for all $\delta^1 \in S, 1 \in R_i(\delta^1|G(S,S'))$ and for all $\tilde{\delta}^1 \in S', 1 \notin R_i(\tilde{\delta}^1|G(S,S'))$.

For any game $G = (u_i, A_i)_{i\in N}$, the mapping $t_{-i} \to R_{-i}(t_{-i}|G)$ defines the set of rationalizable actions for any profile of the opponents' types. For any set A, denote 2^A the set of subsets of A. For any measurable subset $S \subseteq \Delta\Theta \times 2^{A_{-i}}$, let

$$\omega(t_i|G)[S] = \pi_i(t_i)[\{t_{-i} : (\pi_i(t_i, t_{-i}), R_{-i}(t_{-i}|G)) \in S\}].$$

We call $\omega(t_i|G) \in \Delta(\Delta\Theta \times 2^{A_{-i}})$ player i's rationalizable belief at t_i. It is straightforward to see that rationalizable beliefs at types determine the sets of rationalizable conjectures and therefore the sets of best response actions.

If $\delta^2(t_i) \neq \delta^2(t_i')$, the two types must differ in their beliefs at some closed subset $S \subseteq \times_{i\neq j}\Delta(\Delta\Theta)$, thus from above, there must be some pair of disjoint closed subsets $S, S' \subseteq \times_{j\neq i}\Delta(\Delta\Theta)$ and a game $G(S, S')$ that separates them such that $\omega(t_i|G(S,S')) \neq \omega(t_i'|G(S,S'))$. If player i believes the opponents are playing $G(S, S')$, at t_i, t_i' she will have different sets of conjectures about opponents' actions and states of nature; this suggests that she will have different sets of rationalizable actions at t_i and t_i' given that her payoff function is properly designed, as is shown in the following result.

Theorem 2 (Ely and Peski (2006), Theorem 3)**.** *If two types t_i and t_i' differ in terms of their rationalizable belief in game G, i.e., $\omega(t_i|G) \neq \omega(t_i'|G)$, then there is a finite game G' in which t_i and t_i' have distinct rationalizable sets, i.e., $R_i(t_i|G') \neq R_i(t_i'|G')$.*

As an immediate result, if $\delta^2(t_i) \neq \delta^2(t'_i)$, then there is a finite game G' such that $R_i(t_i|G') \neq R_i(t'_i|G')$.

The game G' can also be constructed directly without referring to Theorem 2. The construction of G' is very similar to the construction of $G(\delta^1(t_i), \delta^1(t'_i))$ in step 1. It uses a lemma more general than Lemma 2. Let F be the set of $f : \Delta\Theta \times 2^{A_{-i}} \to [0, \infty)$ such that for any $\beta \in \Delta\Theta, S_j \subseteq A_j, \forall j \neq i$,

$$f(\beta, S_{-i}) = \max_{\substack{\mathbf{k} \in \{1,...,m\}^{N-1} \\ a_{j1},...,a_{jm'} \in S_j, \forall j \neq i}} \beta[\psi(\mathbf{k}, (a_{j,1}, ..., a_{j,m'})_{j \neq i}, \theta)]$$

for some natural numbers m and m', and continuous bounded function $\psi : \{1, ..., m\}^{N-1} \times A_{-i}^{m'} \times \Theta \to [0, \infty)$.

Lemma 3. *The collection of sets* $\{\mu : \mu[f] < 0\} \subseteq \Delta(\Delta\Theta \times 2^{A_{-i}})$*, one for each* $f \in F$*, generate the weak***-topology on* $\Delta(\Delta\Theta \times 2^{A_{-i}})$*. This topology is normal, and therefore any pair of disjoint closed subsets* $S, S' \in \Delta(\Delta\Theta \times 2^{A_{-i}})$ *can be separated by open sets, and there is a function* $f \in F$ *such that* $\forall \mu \in S$ *and* $\mu' \in S'$*,*

$$\mu[f] \neq \mu'[f].$$

As a result of this lemma, there is a game $G(S.S')$ that separates any pair of disjoint closed subsets S, S' of second-order beliefs. In this game, the action set of player i has a product structure, in which the first coordinate of each action represents the action that the opponents are playing against.

The induction works as follows. If $\delta^3(t_i) \neq \delta^3(t'_i)$, the two types must differ in their beliefs at some closed subset $S \in \times_{j \neq i} \Delta(\Delta(\Delta\Theta))$; hence there must be some pair of disjoint closed subsets $S, S' \in \times_{j \neq i} \Delta(\Delta(\Delta\Theta))$ and a game $G(S, S')$ that separate them such that $\omega(t_i|G(S, S')) \neq \omega(t'_i|G(S, S'))$. Applying Theorem 2 again, there must be a finite game G' such that $R_i(t_i|G') \neq R_i(t'_i|G')$.

For $\delta^k(t_i) \neq \delta^k(t'_i), k \geq 3$, respective separating games can be constructed inductively by applying Lemma 3 and Theorem 2. ■

4 Discussions

We can be more explicit about the type-correlated strategy. It takes the form of a partially correlating device, which sends correlated signals (in the canonical form, recommendations) to players depending on their types while preserving players' conditional beliefs. Tang (2015) shows that such correlating devices characterize the correlations embedded among type spaces that have the same set of Δ-hierarchies of beliefs, then points out that belief-invariant Bayesian solution proposed by Forges (1993) is defined using exactly the same correlating devices.

In a closely related study, Liu (2015) characterizes the correlations embedded in type spaces generating the same set of conventional hierarchies of beliefs with state-dependent correlating devices; he also defines a notion of correlated equilibrium with such correlating devices.[9] Essentially, state-dependent correlating devices send correlated signals to players depending on the state of the world while preserving players' beliefs in the original type space, and they capture exactly the same form of correlations permitted in ICR. The comparison between partially correlating devices with state-dependent correlating devices offers us another perspective on the distinction between IPCR and ICR, and similarly, on the distinction between Δ-hierarchy of beliefs and conventional hierarchy of beliefs.

[9] Liu (2009) studies redundant type spaces w.r.t. conventional hierarchies of beliefs from the perspective that every redundant type space can be viewed as a non-redundant type space with an extended space of basic uncertainties.

The payoff equivalence between IPCR and the belief-invariant Bayesian solution is straightforward following the same idea used by Brandenburger and Dekel (1987), who show that in complete information games, correlated rationalizability and a posteriori equilibrium are equivalent. The belief-invariant Bayesian solution is also studied in Forges (2006), Bergemann and Morris (2014), Lehrer et al. (2010) and Lehrer et al. (2013). Among them, Lehrer et al. (2010) and Lehrer et al. (2013) focus on the payoff equivalence between information structures under various solution concepts. The idea behind partially correlating devices resembles that of the non-communicating garblings they use.

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