Hierarchies of Beliefs and the Belief-invariant Bayesian Solution*

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Abstract

The belief-invariant Bayesian solution is a notion of correlated equilibrium in games with incomplete information proposed by Forges (1993), and hierarchy of beliefs over conditional beliefs is introduced by Ely and Peski (2006) in their study of interim independent rationalizability. We study the connection between the two concepts. We partially characterize the correlations embedded among type spaces with the same set of hierarchies of beliefs over conditional beliefs with partially correlating devices, which send correlated signals to players in a way that preserves each player's belief about others' types. Since the belief-invariant Bayesian solution is also implemented by such correlating devices, we then establish that it is invariant on equivalent type space.

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1 Introduction

Harsanyi (1967-1968) proposes type spaces to model players' beliefs and higher-order beliefs in games with incomplete information, and later Mertens and Zamir (1985) construct a universal type space which incorporates all hierarchies of beliefs. These works provide the foundations for strategic analysis of games with incomplete information. One phenomenon that recently attracts game theorists' attention is that, for a given solution concept, type spaces and hierarchies of beliefs are not always strategically equivalent. To be more precise, multiple type spaces can represent the same set of hierarchies of beliefs and are hence equivalent in this respect. However, equivalent type spaces may differ in the amount of correlations incorporated among types, and consequently, for a given solution concept, they may induce different predictions on the actions played by types with the same hierarchy of beliefs.

In particular, type spaces that represent the same set of Mertens-Zamir hierarchies of beliefs always induce the same interim correlated rationalizable outcomes (Dekel et al., 2006; Dekel et al., 2007), but may induce different interim independently rationalizable outcomes (Ely and Peski, 2006) and Bayes Nash equilibrium outcomes.

The implicit correlations, or hidden information, that hierarchies of beliefs fail to incorporate are exactly those embedded among equivalent type spaces. Liu (2015) shows how to distill such correlations and explicitly characterizes them via state-dependent correlating devices. According to the characterization, a type space has the same set of

Mertens-Zamir hierarchies of beliefs as a non-redundant type space if and only if it can be generated from the conjunction of the non-redundant type space and a state-dependent correlating device.[1] Therefore, the characterization decomposes the information in type space into a correlating device and the primitive information–the hierarchies of beliefs.

Hierarchy of beliefs over conditional beliefs, also called Δ-hierarchy of beliefs, is introduced by Ely and Peski (2006) and then extended by Tang (2015) in the study of interim rationalizability. It incorporates richer information than Mertens-Zamir hierarchy of beliefs. As a result, it identifies a stronger notion of solution concept–Ely and Peski show that two types with the same hierarchy of beliefs always have the same set of interim independently rationalizable actions.

In the same spirit of Liu (2015)'s work, we are interested in understanding the correlations embedded among type spaces with the same set of Δ-hierarchies of beliefs and their connection to correlated equilibrium in games with incomplete information. We begin with providing a partial characterization of such correlations. For our purpose, two type spaces are said to be equivalent if they have the same set of Δ-hierarchies of beliefs. A partially correlating device sends correlated signals to players based on interim stage information and preserves each player's belief about others' types. We show that any type space is equivalent to its conjunction with a partially correlating device, and that if two type spaces are equivalent, then they can always generate the same type space via conjunctions with partially correlating devices.

The correlations incorporated in partially correlating devices and the belief-invariant Bayesian solution (Forges, 1993;Forges, 2006) take the same form. When the signals in the correlating device are simply recommendations of actions to the players, the conjunction of the type space and the correlating device generates exactly an epistemic model for the

[1] In a closely related work, Liu (2009) shows that under mild assumptions, a redundant type space can be represented by a non-redundant type space defined on an extended space of basic uncertainties.

belief-invariant Bayesian solution. Based on the partial characterization of correlations obtained, we establish that for any game-form, the set of belief-invariant Bayesian solution payoffs on a type space is the union of its Bayesian Nash equilibria payoffs across equivalent type spaces. And in an immediate corollary, we show that the belief-invariant Bayesian solution is invariant across equivalent type spaces. Therefore, we have obtained a belief foundation for the belief-invariant Bayesian solution. To identify the set of all such solutions, it is without loss of generality for us to consider the universal type space of Δ-hierarchies of beliefs.[2]

This paper is closely related to and is in parallel with Liu (2015). We can find counterparts for various concepts and results of our paper in Liu's work. Since Δ-hierarchy of beliefs incorporates richer information than Mertens-Zamir hierarchy of beliefs, it identifies stronger notions of solution concepts (interim independent rationalizability and the belief-invariant Bayesian solution versus interim correlated rationalizability and Liu's notion of correlated equilibrium), and leaves less information embedded among equivalent type spaces (partially correlating devices versus state-dependent correlating devices). Lehrer et al. (2010) and Lehrer et al. (2013) study the relationship between garblings and the equivalence of type spaces. The non-communicating garblings that they use have similar features as partially correlating devices. Bergemann and Morris (2014) propose and study the Bayes correlated equilibrium. They focus on correlating devices that are dependent on ex post stage information but free from belief-invariance restrictions.

The rest of this paper is organized as follows. We present the basic notations Section 2, and formulate hierarchies of beliefs in Section 3. In Section 4, we characterize the correlations embedded in equivalent type spaces and then apply the result on the belief-

[2] An application is on the robustness of solution concepts to incomplete information, a study initiated by Kajii and Morris (1997). The belief foundation of the belief-invariant Bayesian solution allows us to study its robustness by perturbing the Δ-hierarchy of beliefs of a type, using the interim approach taken by Weinstein and Yildiz (2007) in their study of the robustness of interim correlated rationalizability.

invariant Bayesian solution. Section 5 concludes.

2 Notations

For any metric space X, let ΔX denote the space of probability measures on the Borel σ-algebra of X endowed with the weak*-topology. Let the product of two metric spaces be endowed with the product Borel σ-algebra. For any probability measure $\mu \in \Delta X$, let $\operatorname{supp} \mu$ denote the support of μ, and for any measure $\mu \in \Delta(X \times Y)$, let $\operatorname{marg}_X \mu$ denote the marginal of μ on X.

We study games with incomplete information with n players. The set of players is $N = \{1, 2, ..., n\}$. For each $i \in N$, let $-i$ denote the set of i's opponents. Players play a game in which the payoffs are uncertain and parameterized by a finite set Θ. Each element $\theta \in \Theta$ is called a state of nature. For each $i \in N$, let A_i be the set of actions for player i and let $A \equiv \times_{i \in N} A_i$ be the set of action profiles. A (strategic-form) game is a profile $G = (g_i, A_i)_{i \in N}$. For each $i \in N$, we assume the payoff function is bounded: $g_i : A \times \Theta \to [-M, M]$, for some positive real number M. The set of finite bounded games is denoted by $\mathcal{G}$.

A type space over Θ is defined as $T = (T_i, \pi_i)_{i \in N}$, where for each i, T_i is a finite set of types for player i and $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$ is a belief mapping such that $\pi_i(t_i)[(t_{-i}, \theta)]$ describes player i's belief on the event that the others' type profile is t_{-i} and the state of nature is θ.[3]

[3]Throughout, given arbitrary $x \in X$ and $y \in Y$, we use the notation $\pi_i(x)[y]$ to denote player i's belief about y conditional on x. More precisely, the object in the round bracket always denotes the object player that i conditions on, and the object in the square bracket always denotes the object that i assigns probability to.

3 Hierarchies of beliefs

3.1 Formulation

We first present Mertens and Zamir's standard formulation of hierarchies of beliefs (see also Brandenburger and Dekel, 1993), and based on that present Ely and Peski's construction of Δ-hierarchies of beliefs.

Let $X_0 = \Theta$, and for each $k \geq 1$, let $X_k = X_{k-1} \times (\Delta(X_{k-1}))^{n-1}$. Let $h^1(t_i) = \text{marg}_\Theta\, \pi_i(t_i)$ be player i's belief over Θ at type t_i. For each $k \geq 2$, let $h^k(t_i)[S] = \pi_i(t_i)[\{(\theta, t_{-i}) : (\theta, (h^l(t_{-i}))_{1\leq l\leq k-1}) \in S\}]$, for any measurable subset $S \subseteq X_{k-1}$. In this construction, $h^k(t_i) \in \Delta(X_{k-1})$ represents player i's k-th order belief at t_i.

The profile $h(t_i) = (h^1(t_i), ..., h^k(t_i), ...) \in \times_{k=0}^{\infty} \Delta X_k$ is called player i's Mertens-Zamir hierarchy of beliefs at type t_i.

As extensions of Mertens-Zamir hierarchies of beliefs, Δ-hierarchies of beliefs describe players' beliefs and higher-order beliefs about the conditional beliefs on states of nature. This concept is introduced by Ely and Peski (2006) in their study of interim independent rationalizability. We begin with defining conditional beliefs. Given belief $\pi_i(t_i) \in \Delta(T_{-i} \times \Theta)$, the conditional belief of type t_i over Θ, conditioning on the others' types being t_{-i}, is denoted by $\pi_i(t_i)(t_{-i}) \in \Delta(\Theta)$, also written as $\pi_i(t_i, t_{-i})$. For any type space T, let $B_i(t_i) = \{\pi_i(t_i, t_{-i}) \in \Delta(\Theta) : t_{-i} \in T_{-i}\}$ be the set of all possible conditional beliefs at t_i. Type t_i's belief over T_{-i} then induces a belief over $B_i(t_i) \subset \Delta(\Theta)$: for any measurable subset $S \subseteq \Delta(\Theta)$, $\pi_i(t_i)[S] = \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) \in S\}]$.

The definition of Δ-hierarchy of beliefs at t_i treats the set of possible conditional beliefs, i.e., $\Delta(\Theta)$, as the set of basic uncertainties. In a Δ-hierarchy of beliefs, the first-order belief of a player is her belief over the set of conditional beliefs, the second-order belief is her belief over the others' beliefs over the set of conditional beliefs, and so on.

For any type space $T = (T_i, \pi_i)_{i \in N}$ on Θ, we can transform it into a type space $T^\Delta = (T_i, \pi_i^\Delta)_{i \in N}$ on $\Delta(\Theta)$. In the new type space, players' types are unchanged, and player i's belief at type t_i is $\pi_i^\Delta(t_i) \in \Delta(T_{-i} \times \Delta(\Theta))$, such that

$$\pi_i^\Delta(t_i)[S] = \pi_i(t_i)[\{t_{-i} : (t_{-i}, \pi_i(t_i, t_{-i})) \in S\}],$$

for any measurable subset $S \subseteq T_{-i} \times \Delta(\Theta)$.

In the type space T^Δ on $\Delta(\Theta)$, let the Mertens-Zamir hierarchy of beliefs at t_i be denoted by $h(t_i|T^\Delta)$.

Definition 1. *For any type space T and integer $k \geq 1$, let the k-th order Δ-hierarchy of beliefs at $t_i \in T_i$ be $h^k(t_i|T^\Delta)$ and denote it by $\delta^k(t_i)$. Also, let $\delta(t_i) = (\delta^1(t_i), ..., \delta^k(t_i), ...)$ denote the Δ-hierarchy of beliefs at t_i.*

By definition, $\delta(t_i) = h(t_i|T^\Delta)$. For player i, we use δ_{-i} to denote the profile of the others' Δ-hierarchies of beliefs.

3.2 Equivalence of type spaces

For any type space T, let the set of all Δ-hierarchies of beliefs of player i be $\Lambda_i(T) = \{\delta(t_i) : t_i \in T_i\}$. Just like Mertens-Zamir hierarchies of beliefs, the set of Δ-hierarchies of beliefs does not uniquely pin down a type space. Instead, multiple type spaces may induce the same set of Δ-hierarchies of beliefs.

Definition 2. *Two type spaces T and T' are equivalent, written as $T \sim T'$, if they have the same set of Δ-hierarchies of beliefs for all players, that is, if*

$$\Lambda_i(T) = \Lambda_i(T'), \forall i \in N.$$

A type space in which different types of a player always have different hierarchies of beliefs is called a reduced type space (Aumann, 1998), or a non-redundant type space (Mertens and Zamir, 1985). For any Mertens-Zamir hierarchy of beliefs, we are able to construct such a type space that generates it. However, this is not true for Δ-hierarchies of beliefs. We illustrate this with a simple type space taken from Ely and Peski (2006).

Example 1. Consider a type space T in which $\Theta = T_1 = T_2 = \{+1, -1\}$, and players' beliefs are updated from a common prior $\pi \in \Delta(\Theta \times T_1 \times T_2)$ such that

$$\pi(t_1, t_2, \theta) = \begin{cases} \frac{1}{4} \text{ if } t_i \cdot t_2 = \theta, \\ 0 \text{ otherwise.} \end{cases}$$

In this type space, the set of conditional beliefs for each type contains point mass on $\theta = +1$ and point mass on $\theta = -1$, and both types of player 1 (or player 2) have the same Δ-hierarchy of beliefs–common certainty of equal probability on the point masses. However, type space T is the most compact one that supports this Δ-hierarchy of beliefs.

4 Main results

4.1 A partial characterization of correlations

Without distinguishing non-redundant and redundant type spaces, we can achieve a partial characterization of the correlation embedded across equivalent type spaces. As we shall see, this partial characterization is sufficient for the invariance of the belief-invariant solution.

Definition 3. *For any type space T, a partially correlating device on T is a profile $Q = (q_i, S_i)_{i \in N}$, where for each $i \in N, S_i$ is a finite set of signals and $q_i : T \rightarrow \Delta S,$*

where $S = \times_{i \in N} S_i$, is a belief mapping such that

1. *For any $i \neq j, t \in T, \operatorname{supp} q_i(t) = \operatorname{supp} q_j(t)$.*

2. *belief invariance is satisfied: at different types t_{-i}, t'_{-i} of the others', player i receives s_i with the same probability, i.e.,*

$$\sum_{\{s' \in S : s'_i = s_i\}} q_i(t_i, t_{-i})[s'] = \sum_{\{s' \in S : s'_i = s_i\}} q_i(t_i, t'_{-i})[s'], \forall i, t_i, s_i.$$

From the definition, player i believes that at $(t_i, t_{-i}) \in T$, the partially correlating device selects a signal profile $(s_i, s_{-i}) \in S$ according to the distribution $q_i(t_i, t_{-i}) \in \Delta S$, and for each $j \in N$, s_j will be reported by a mediator to player j. Belief invariance ensures that from the signals that the players receive, they cannot infer any extra information about the others' types. Also note that the correlated signals depend only on the interim stage information–players' types–not on states of nature.

The partially correlating device here and the state-dependent correlating device of Liu (2015) take similar forms, while differing from each other mainly in two ways: (i) in the former, correlated signals are dependent on the interim stage information (the type profile of players), while in the latter, they are dependent on the ex post stage information (the state of the world, i.e., both the type profile of players and the true state of nature); and (ii) the belief-invariance restriction of the former requires that the signals do not change each player's belief about the others' types, while that of the latter requires that they do not change each player's belief about the states of the world. Every partially correlating device is a state-dependent correlating device, but the converse is not true.

When the partially correlating device uses actions as signals and therefore the signals are simply recommendations of play, we say that the correlating device is canonical.

Definition 4. *If for all $i \in N, S_i = A_i$, then Q is a canonical partially correlating device.*

Let $q_i(t_i, t_{-i})[s_{-i}|s_i]$ be player i's belief on the others' receiving the signal profile s_{-i}, conditional on her receiving s_i. This conditional belief can be easily calculated given $q : T \to \Delta S$.

Definition 5. *For any type space $T = (T_i, \pi_i)_{i \in N}$ and any partially correlating device $Q = (q_i, S_i)_{i \in N}$, let T^Q be the enlarged type space generated from T through operating Q on T. More precisely, $T^Q = (T_i^Q, \pi_i^Q)_{i \in N}$ such that*

$$T_i^Q = \{(t_i, s_i) : t_i \in T_i, q_i(t)[s_i] > 0, \text{ for some } t_{-i} \in T_{-i}\},$$

and for all $(t_{-i}, s_{-i}) \in T_{-i}^Q, \theta \in \Theta$ and $(t_i, s_i) \in T_i^Q$,

$$\pi_i^Q((t_i, s_i))[((t_{-i}, s_{-i}), \theta)] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot q_i(t_i, t_{-i})[s_{-i}|s_i].$$

Intuitively, we can view T^Q as the type space faced by players at the ex ante stage, when in addition a correlating device is expected to send signals to players after the realization of players' types.

The following theorem provides a partial characterization of the correlation embedded in equivalent type spaces.

Proposition 1. *Let T be any type space. Then*

1. *for any partially correlating device Q, $T^Q \sim T$; more specifically, for any $(t_i, s_i) \in T_i^Q, \delta((t_i, s_i)) = \delta(t_i)$.*

2. *for any type space $\hat{T}$ satisfying $T \sim \hat{T}$, there exist partially correlating devices Q and $\hat{Q}$ such that $T^Q = \hat{T}^{\hat{Q}}$.*

The first part of the proposition states that the enlarged type space T^Q generated by type space T and any partially correlating device Q always has the same set of Δ-

hierarchies of beliefs as T does. In particular, after receiving any signal s_i at type t_i, player i's Δ-hierarchy of beliefs does not change. The second part of the proposition states that for any pair of type spaces T and $\hat{T}$ which have the same set of Δ-hierarchies of beliefs, we can find two partially correlating device Q and $\hat{Q}$ through which T and $\hat{T}$ generate the same enlarged type space. Our characterization of correlation is partial because unlike Liu (2015), we do not show that any redundant type space can be enlarged from the non-redundant type space with a correlating device. Instead, we establish the connection between two equivalent type spaces without distinguishing redundant and non-redundant type spaces.

The proof of this proposition is relegated into Appendix A. The proof of part 1 is by induction, and the key is the belief invariance property of partially correlating devices. Next, we briefly sketch the intuition for the proof of part 2. We transform the type space $\hat{T}$ into a partially correlating device Q and transform the type space T into a partially correlating device $\hat{Q}$. Then we show that T^Q and $\hat{T}^{\hat{Q}}$ both represent some version of the product type space $T \times \hat{T}$ and hence are the same. Specifically, when the correlating device Q is operated on type space T, from player i's view, at any type profile (t_i, t_{-i}), the set of signals she may receive from Q is exactly the set of types in the type space $\hat{T}$ that have the same Δ-hierarchy of beliefs as t_i, and when she receives any such signal $\hat{t}_i$, the set of profiles of signals that her opponents may receive from Q is exactly the set of type profiles $\hat{t}_{-i}$ in $\hat{T}_{-i}$ that induce the same profile of Δ-hierarchies as t_{-i} and the same conditional belief at $\hat{t}_i$ as $\pi_i(t_i, t_{-i})$.

4.2 The belief-invariant Bayesian solution

The belief-invariant Bayesian solution is a notion of correlated equilibrium for games with incomplete information proposed by Forges (1993)[4] and later discussed in Forges (2006) and Bergemann and Morris (2014). A belief-invariant Bayesian solution on a type space T is defined as an epistemic model that embeds T in a belief-invariant way and satisfies Bayesian rationality at each state of the world. As Forges (2006) points out, each such epistemic model is an enlarged type space T^Q for some partially correlating device Q, and each belief-invariant solution corresponds to a Bayes Nash equilibrium (BNE) of the game (G, T^Q).[5]

Therefore, the set of belief-invariant Bayesian solution payoffs on T is the union of BNE payoffs across enlarged type spaces.

Let $\mathcal{Q}$ denote the set of all partially correlating devices. For each game (G, T) and a BNE σ of this game, let $g_i(t_i|\sigma)$ be player i's interim payoff at type t_i, and let $g(t|\sigma)$ be the interim payoff vector at type profile $t \in T$. Denote the set of interim BNE payoffs of the game (G, T) as

$$NE(G, T) = \{g(t|\sigma) : t \in T, \sigma \text{ is a BNE of } (G, T)\}.$$

Let $B_I(G, T)$ denote the set of interim belief-invariant Bayesian solution payoffs, then

$$B_I(G, T) = \cup_{\{Q:Q \in \mathcal{Q}\}} NE(G, T^Q).$$

The result below states that the set of players' interim payoffs from belief-invariant Bayesian solutions at a type space is exactly the union of interim BNE payoffs on equiva-

[4]Forges' definition of the Bayesian solution is restricted to two-player games for type spaces with common priors; what we present here is the n-player non-common prior analogue of her definition.

[5]From a point of view analogous to the "revelation principle" in the mechanism design literature, Forges (2006) also characterizes belief-invariant Bayesian solution payoffs with canonical partially correlating devices.

lent type spaces.

Proposition 2. $B_I(G,T) = \cup_{\{\hat{T}:\hat{T}\sim T\}} NE(G,\hat{T})$.

Proof. By Proposition 1, for all $Q \in \mathcal{Q}, T \sim T^Q$. Therefore, it is straightforward that $\bigcup_{\{Q:Q\in\mathcal{Q}\}} NE(G,T^Q) \subset \bigcup_{\{\hat{T}:\hat{T}\sim T\}} NE(G,\hat{T})$. We also know that for any $\hat{T}$ such that $\hat{T} \sim T$, there exist $\hat{Q}$ and Q such that $\hat{T}^{\hat{Q}} = T^Q$. Therefore,

$$NE(G,\hat{T}) \subset NE(G,\hat{T}^{\hat{Q}}) \Rightarrow NE(G,\hat{T}) \subset NE(G,T^Q).$$

That is, we also have $\bigcup_{\{\hat{T}:\hat{T}\sim T\}} NE(G,\hat{T}) \subset \bigcup_{\{Q:Q\in\mathcal{Q}\}} NE(G,T^Q)$.

□

Lehrer et al. (2010) and Lehrer et al. (2013) study the payoff equivalence of information structures with respect to belief-invariant Bayesian solutions. They show that two information structures are equivalent if and only if there exist non-communicating garbling transformations between them. Non-communicating garblings share similar features as partially correlating devices. The main difference between them is that a garbling garbles original signals, while a correlating device introduces new signals into the structure.

It is immediate from Proposition 2 that if two type spaces represent the same set of Δ-hierarchies of beliefs, they must induce the same set of belief-invariant Bayesian solution payoffs in any game. In other words, the belief-invariant Bayesian solution is invariant on the equivalent class of type spaces.

Corollary 1. *If $\hat{T} \sim T$, then $B_I(G,T) = B_I(G,\hat{T})$.*

5 Conclusion

We study the correlations embedded in type spaces with the same set of hierarchies of beliefs over conditional beliefs, it turns out that such correlations can be expressed explicitly with partially correlating devices, which operate in the interim stage of the game.

With these results, we see a complete picture of the connections among Δ-hierarchies of beliefs, interim partially correlated rationalizability, and the belief-invariant Bayesian solution. We already know that Partially correlating devices characterize correlations embedded in type spaces with the same set of Δ-hierarchies of beliefs, and implement the Bayesian solution. Furthermore, Tang (2015) shows that Δ-hierarchies of beliefs fully identify interim partially correlated rationalizability. As a result, interim partially correlated rationalizability and the belief-invariant Bayesian solution are payoff equivalent.

A Appendix: Proof of Proposition 1

Part I. We use induction to show that for any $(t_i, s_i) \in T_i^Q, \delta((t_i, s_i)) = \delta(t_i)$. First note that for any $(t_i, s_i) \in T_i^Q$, $(t_{-i}, s_{-i}) \in T_{-i}^Q$, and $\theta \in \Theta$,

$$\begin{aligned}
\pi_i^Q((t_i, s_i), (t_{-i}, s_{-i}))[\theta] &= \frac{\pi_i^Q((t_i, s_i))[(t_{-i}, s_{-i}), \theta]}{\pi_i^Q((t_i, s_i))[(t_{-i}, s_{-i})]} \\
&= \frac{\pi_i(t_i)[(t_{-i}, \theta)] \cdot q_i(t_i, t_{-i})[(s_i, s_{-i})]}{\pi_i(t_i)[t_{-i}] \cdot q_i(t_i, t_{-i})[(s_i, s_{-i})]} \\
&= \pi_i(t_i, t_{-i})[\theta].
\end{aligned}$$

Therefore, for any $(t_i, s_i) \in T_i^Q$, the set of conditional beliefs at (t_i, s_i) is the same as that at t_i. Furthermore, for any conditional belief $\beta \in B_i(t_i)$,

$$\begin{aligned}
\pi_i^Q((t_i,s_i))[\beta] &= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i^Q((t_i,s_i),(t_{-i},s_{-i})) = \beta\}] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i^Q((t_i,s_i))[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta\}] \\
&= \pi_i(t_i)[\beta].
\end{aligned}$$

The fourth equation above comes from belief invariance. We have proved that for all $(t_i,s_i) \in T_i^Q$, $\delta^1((t_i,s_i)) = \delta^1(t_i)$. For higher-order beliefs, we prove by induction. Now suppose for all $0 < l \leq k$ and $(t_i,s_i) \in T_i^Q, \delta^l((t_i,s_i)) = \delta^l(t_i)$, we show that for all $(t_i,s_i) \in T_i^Q$, $\delta^{k+1}((t_i,s_i)) = \delta^{k+1}(t_i)$. Let the support of the l-th order belief at type t_i be $B_i^l(t_i)$. As a result, the set of conditional beliefs is relabeled as $B_i^1(t_i)$. By the premises of induction, for all $(t_i,s_i) \in T_i^Q$ and $0 < l \leq k, B_i^l((t_i,s_i)) = B_i^l(t_i)$. Indeed, for any $(\beta,\delta^1,...,\delta^k) \in \times_{0<l\leq k} B_i^l(t_i)$,

$$\begin{aligned}
&\delta^{k+1}((t_i,s_i))[(\beta,\delta^1,...,\delta^k)] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : (\pi_i^Q((t_i,s_i),(t_{-i},s_{-i})),(\delta^l((t_{-i},s_{-i})))_{l=1}^k) = (\beta,\delta^1,\ldots\delta^k)\}] \\
&= \pi_i^Q((t_i,s_i))[\{(t_{-i},s_{-i}) : \pi_i(t_i,t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1,...,\delta^k(t_{-i}) = \delta^k\}] \\
&= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i,t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1,...,\delta^k(t_{-i}) = \delta^k\}] \\
&= \delta^{k+1}(t_i)[(\beta,\delta^1,...,\delta^k)].
\end{aligned}$$

By induction, for all $(t_i,s_i) \in T_i^Q, \delta((t_i,s_i)) = \delta(t_i)$. Naturally, T^Q and T have the same set of Δ-hierarchies of beliefs, $T^Q \sim T$.

Part II. Fix a pair of type spaces $T = (T_i,\pi_i)_{i\in N}$ and $\hat{T} = (\hat{T}_i,\hat{\pi}_i)_{i\in N}$. Suppose $T \sim \hat{T}$, we now construct Q and $\hat{Q}$ such that $T^Q = \hat{T}^{\hat{Q}}$. To do that, we transform the type space $\hat{T}$ into a partially correlating device Q and transform T into a partially correlating device $\hat{Q}$. We then show that the generated type spaces T^Q and $\hat{T}^{\hat{Q}}$ are the same.

Step 1. Before we start, we need a few intermediate results. The lemma below says that if two types have the same Δ-hierarchy of beliefs, then they must have the same belief over conditional beliefs and the others' Δ-hierarchies of beliefs.

Lemma 1. *Fix type spaces T and T'. If $t_i \in T_i, t'_i \in T'_i$ and $\delta(t_i) = \delta(t'_i)$, then $\pi_i(t_i)[(\beta, \delta_{-i})] = \pi'_i(t'_i)[(\beta, \delta_{-i})], \forall \beta, \delta_{-i}$.*

Proof. With the basic property of probability measures,

$$\begin{aligned}
\pi_i(t_i)[(\beta, \delta_{-i})] &= \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}, ...\}] \\
&= \pi_i(t_i)[\cap_n \{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}\}] \\
&= \lim_n \pi_i(t_i)[\{t_{-i} : \pi_i(t_i, t_{-i}) = \beta, \delta^1(t_{-i}) = \delta^1_{-i}, ..., \delta^n(t_{-i}) = \delta^n_{-i}\}] \\
&= \lim_n \delta^{n+1}(t_i)[(\beta, \delta^1, ..., \delta^n)] \\
&= \lim_n \delta^{n+1}(t'_i)[(\beta, \delta^1, ..., \delta^n)] \\
&= \pi'_i(t'_i)[(\beta, \delta_{-i})].
\end{aligned}$$

□

Also, if two types $t_i \in T, t'_i \in T'$ have the same Δ-hierarchy of beliefs, and if t_i has a conditional belief $\pi_i(t_i, t_{-i})$, then t'_i must also have such a conditional belief, conditional on some type profile of the others t'_{-i} which have the same Δ-hierarchies of beliefs as t_{-i}.

Lemma 2. *Fix type spaces T and T'. Suppose $t_i \in T_i, t'_i \in T'_i$, and $\delta(t_i) = \delta(t'_i)$. Then for any $t_{-i} \in T_{-i}$ that satisfies $\pi_i(t_i)[t_{-i}] > 0$, there exists $t'_{-i} \in T'_{-i}$ that satisfies $\delta(t'_{-i}) = \delta(t_{-i})$ and $\pi'_i(t'_i)[t'_{-i}] > 0$, such that $\pi_i(t_i, t_{-i}) = \pi'_i(t'_i, t'_{-i})$.*

Proof. We prove by contradiction. Suppose it is not the case. Then there exists a t_{-i} that satisfies $\pi_i(t_i)[t_{-i}] > 0$ and $\pi_i(t_i, t_{-i}) = \beta$, such that for all t'_{-i} that satisfies $\pi'_i(t'_i, t'_{-i}) = \beta$, $\pi'_i(t'_i)[t'_{-i}] > 0$, it must be that $\delta(t'_{-i}) \neq \delta(t_{-i})$. As a result, $\pi'_i(t'_i)[(\beta, \delta_{-i}(t_{-i}))] = 0$. However, $\pi_i(t_i)[(\beta, \delta_{-i}(t_{-i}))] \geq \pi_i(t_i)[t_{-i}] > 0$. Given Lemma 1, this is in contradiction with $\delta(t_i) = \delta(t'_i)$.

□

Step 2. Using information in type space $\hat{T}$, we now construct a partially correlating device $Q = (q_i, S_i)_{i \in N}$ which is to be operated on type space T. For each $i \in N$, let the set of signals for player i be $S_i = \hat{T}_i$, and define $S \equiv \times_{i \in N} S_i$. Define

$$S_i(t_i) \equiv \{\hat{t}_i \in \hat{T}_i : \delta(\hat{t}_i) = \delta(t_i)\}$$

and

$$S_{-i}(t_i, t_{-i}|\hat{t}_i)) \equiv \{\hat{t}_{-i} \in \hat{T}_{-i} : \delta(\hat{t}_{-i}) = \delta(t_{-i}) \text{ and } \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i}) = \pi_i(t_i, t_{-i})\}.$$

Intuitively, we are going to construct $q_i : T \to \Delta S$ in a way such that the set of signals that player i could possibly receive when her type is t_i is restricted to be $S_i(t_i)$, which is the set of t_i's equivalent types in $\hat{T}_i$. Similarly, $S_{-i}(t_i, t_{-i}|\hat{t}_i))$ will be the restricted set of signals $\hat{t}_{-i}$ that players $-i$ may receive at type profile t_{-i} from player i's view, when her own type is t_i and she receives signal $\hat{t}_i$.

We need the following result, which is immediate from Lemma 1 and Lemma 2, in the construction of q_i. It states that if two types are equivalent in $\hat{T}$, then their beliefs over conditional beliefs and the others' Δ-hierarchies of beliefs must be the same.

Lemma 3. *If $\hat{t}_i, \hat{u}_i \in S_i(t_i)$, then $\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i))] = \hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]$.*

Define on the type space $\hat{T}$ a prior $\hat{p}_i \in \Delta(\hat{T}_i \times \hat{T}_{-i} \times \Theta)$ for player i as follows:

$$\hat{p}_i[(\hat{t}_i, \hat{t}_{-i}, \theta)] = \frac{1}{|\hat{T}_i|}\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)], \forall (\hat{t}_i, \hat{t}_{-i}, \theta) \in \hat{T}_i \times \hat{T}_{-i} \times \Theta.$$

From player i's view, the partially correlating device operates only in states of the world $(\hat{t}_i, \hat{t}_{-i}, \theta)$ such that $\hat{p}_i(\hat{t}_i, \hat{t}_{-i}, \theta) > 0$. For each $i \in N$, we can construct the belief system $q_i : T \to \Delta S$ as follows:

$$q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] = \begin{cases} \frac{\hat{p}_i[(\hat{t}_i, \hat{t}_{-i})]}{\hat{p}_i[S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)]}, & \text{if } (\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i); \\ 0, & \text{otherwise.} \end{cases}$$

With Lemma 3, for any $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$,

$$\begin{aligned} q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] &= \frac{\hat{p}_i[\hat{t}_i]\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)]}{\sum_{\hat{u}_i \in S_i(t_i)} \hat{p}_i[\hat{u}_i]\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i})|\hat{u}_i]} \\ &= \frac{1/|\hat{T}_i|}{1/|\hat{T}_i| \cdot |S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i})|\hat{t}_i]}. \end{aligned}$$

The expression of q_i can be rewritten as

$$q_i(t_i, t_{-i})[(\hat{t}_i, \hat{t}_{-i})] = \begin{cases} \frac{1}{|S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i)]}, & \text{if } (\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i); \\ 0, & \text{otherwise.} \end{cases}$$

Now we prove that the Q defined above is indeed a partially correlating device. First, for any $i \neq j, t \in T$,

$$\operatorname{supp} q_i(t) = \operatorname{supp} q_j(t) = \times_{k \in N} S_k(t_k).$$

This is because from player i's view, each $\hat{t}_i \in S_i(t_i)$ is sent to her with probability $\frac{1}{|S_i(t_i)|}$, and that for each $\hat{t}_{-i} \in \times_{k \in N \setminus \{i\}} S_k(t_k)$, there must be $\hat{t}_i \in S_i(t_i)$ such that $\hat{t}_{-i} \in$

$S_{-i}(t_i, t_{-i}|\hat{t}_i)$ and $\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] > 0$, due to Lemma 2.

Second, belief invariance is satisfied: for any $(t_i, t_{-i}) \in T_i$ and any $\hat{u}_i \in S_i(t_i)$, the probability that player i will receive signal $\hat{u}_i$ is

$$\begin{aligned}\sum_{\{\hat{t}\in\hat{T}:\hat{t}_i=\hat{u}_i\}} q_i(t_i, t_{-i})[(\hat{u}_i, \hat{t}_{-i})] &= \sum_{\{\hat{t}_{-i}:\hat{t}_{-i}\in S_{-i}(t_i,t_{-i}|\hat{u}_i)\}} \frac{1}{|S_i(t_i)|} \cdot \frac{\hat{\pi}_i(\hat{u}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\ &= \frac{1}{|S_i(t_i)|} \frac{\sum_{\{\hat{t}_{-i}:\hat{t}_{-i}\in S_{-i}(t_i,t_{-i}|\hat{u}_i)\}} \hat{\pi}_i(\hat{u}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\ &= \frac{1}{|S_i(t_i)|} \frac{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]}{\hat{\pi}_i(\hat{u}_i)[S_{-i}(t_i, t_{-i}|\hat{u}_i)]} \\ &= \frac{1}{|S_i(t_i)|},\end{aligned}$$

which is independent of t_{-i}. Thus the signal does not provide extra information on the others' types.

Step 3. Given the partially correlating device Q constructed using information in $\hat{T}$, we can generate a new type space $T^Q = (T_i^Q, \pi_i^Q)_{i\in N}$ from the type space T. In T^Q, $T_i^Q = \{(t_i, \hat{t}_i) : t_i \in T_i, \hat{t}_i \in S_i(t_i)\}$, and for any $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$,

$$\pi_i^Q((t_i, \hat{t}_i))[((t_{-i}, \hat{t}_{-i}), \theta)] = \pi_i(t_i)[(t_{-i}, \theta)] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[S_{-i}(t_i, t_{-i}|\hat{t}_i)]}.$$

Similarly, we can construct another partially correlating device $\hat{Q}$ using information in the type space T, and generate a new type space $\hat{T}^{\hat{Q}}$ from $\hat{T}$. In $\hat{T}^{\hat{Q}}$, $\hat{T}_i^{\hat{Q}} = \{(\hat{t}_i, t_i) : \hat{t}_i \in \hat{T}_i, t_i \in S_i(\hat{t}_i)\}$, and for any $(t_i, t_{-i}) \in S_i(\hat{t}_i) \times S_{-i}(\hat{t}_i, \hat{t}_{-i}|t_i)$,

$$\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)] = \hat{\pi}_i(\hat{t}_i)[(\hat{t}_{-i}, \theta)] \cdot \frac{\pi_i(t_i)[t_{-i}]}{\pi_i(t_i)[S_{-i}(\hat{t}_i, \hat{t}_{-i}|t_i)]}.$$

It is straightforward that $T_i^Q = \hat{T}_i^{\hat{Q}}, \forall i \in N$. Now we show $\pi_i^Q((t_i, \hat{t}_i)) = \pi_i^{\hat{Q}}((\hat{t}_i, t_i))$. By the definition, for any (t_i, t_{-i}) and $(\hat{t}_i, \hat{t}_{-i}) \in S_i(t_i) \times S_{-i}(t_i, t_{-i}|\hat{t}_i)$, we know that

$\pi_i(t_i, t_{-i}) = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i}) = \beta, \delta(t_{-i}) = \delta(\hat{t}_{-i}) = \delta_{-i}$, for some β and δ_{-i}. We can decompose the belief π_i^Q as follows:

$$\begin{aligned}
&\pi_i^Q((t_i, \hat{t}_i))[((t_{-i}, \hat{t}_{-i}), \theta)] \\
&= \pi_i(t_i, t_{-i})[\theta] \cdot \pi_i(t_i)[t_{-i}] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\hat{\pi}_i(\hat{t}_i)[\{\hat{t}'_{-i} : \delta(\hat{t}'_{-i}) = \delta(t_{-i}), \hat{\pi}_i(\hat{t}_i, \hat{t}'_{-i}) = \pi_i(t_i, t_{-i})\}]} \\
&= \pi_i(t_i, t_{-i})[\theta] \cdot \frac{\pi_i(t_i)[t_{-i}] \cdot \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}]}{\pi_i(t_i)[(\beta, \delta_{-i})]}.
\end{aligned}$$

Similarly, $\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)]$ can also be decomposed:

$$\pi_i^{\hat{Q}}((\hat{t}_i, t_i))[((\hat{t}_{-i}, t_{-i}), \theta)] = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i})[\theta] \cdot \frac{\hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] \cdot \pi_i(t_i)[t_{-i}]}{\hat{\pi}_i(\hat{t}_i)[(\beta, \delta_{-i})]}.$$

We compare π_i^Q and $\pi_i^{\hat{Q}}$ term by term. First, $\pi_i(t_i, t_{-i})[\theta] = \hat{\pi}_i(\hat{t}_i, \hat{t}_{-i})[\theta]$. Second, $\pi_i(t_i)[t_{-i}] \cdot \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] = \hat{\pi}_i(\hat{t}_i)[\hat{t}_{-i}] \cdot \pi_i(t_i)[t_{-i}]$. Third, from Lemma 1, $\pi_i(t_i)[(\beta, \delta_{-i})] = \hat{\pi}_i(\hat{t}_i)[(\beta, \delta_{-i})]$.

Since for any $i \in N$, $(t_i, \hat{t}_i) \in T_i^Q = \hat{T}_i^{\hat{Q}}$, $\pi_i^Q((t_i, \hat{t}_i)) = \pi_i^{\hat{Q}}((\hat{t}_i, t_i))$, we have $T^Q = \hat{T}^{\hat{Q}}$.

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